12. ARROW-DEBREU MODEL OF GENERAL EQUILIBRIUM UNDER UNCERTAINTY

Usual insurance contract: Insured person pays premium $pX$ in advance; company pays indemnity $X$ if loss occurs, nothing otherwise

Equivalent alternative: Contract written in advance but no payments made in advance
- Insured pays company $pX$ if loss does not occur (in state 1)
- Company pays insured $(1-p)X$ if loss occurs (in state 2)

The trade (contract) made in advance is merely an exchange of promises
- Need governance mechanism for credibility, but otherwise no problem

Hence more general idea of “trade in contingent claims”
- Like betting slips – promises to pay specified money amounts or deliver specified goods
  if some specified state(s) of the world is(are) realized, and nothing otherwise
- Can pay a sure price in advance, or exchange it for another promise of equal market value

Examples:
[1] Betting on sports events, racing bets, etc.
[2] Betting on outcomes of political and economic events:
   Iowa electronic markets: [http://www.biz.uiowa.edu/iem/](http://www.biz.uiowa.edu/iem/)
DEMAND FOR INSURANCE

Objects traded are slips of paper that promise

$S_1 : \text{"$1 if state 1"}, \ S_2 : \text{"$1 if state 2"}$

Trading occurs before uncertainty is resolved

Prices $p_1$ for one slip $S_1$ ; $p_2$ for one slip $S_2$

Traders price-takers; probability of state 2 is $\pi$

Risk-averse insured person: will have wealth

$W_0$ in state 1 (no-loss), $W_0 – L$ in state 2 (loss)

(so $p_2$ is like premium for $\$1$ of indemnity)

Equivalently, has endowments of

$W_0$ of $S_1$-slips, $W_0 – L$ of $S_2$-slips

Wants to sell $X_1$ of $S_1$ slips, buy $X_2$ of $S_2$ slips

Budget constraint $p_1 \ X_1 – p_2 \ X_2 = 0$

if trade in these markets must be balanced

(Imbalance corresponds to saving or dissaving; will allow later.)

Objective: $EU = (1-\pi) \ U(W_0 – X_1) + \pi \ U( W_0 – L + X_2)$

FOCs: $(1-\pi) \ U'(W_0 – X_1) = \lambda \ p_1 , \ \pi \ U'( W_0 – L + X_2) = \lambda \ p_2$

Risk-neutral insurance company that sells $S_2$ slips has expected profit = $p_2 – \pi$ on each slip

Competition achieves zero profit: $p_2 = \pi$. Similarly, $p_1 = 1 - \pi$

Then FOCs become $U'(W_0 – X_1) = \lambda , \ U'( W_0 – L + X_2) = \lambda$ so full insurance
ARBITRAGE

Can have markets in the $S_1$, $S_2$ slips that pay $1$ in one state, nothing in the other
Can also have a combo slip $S_c$ that pays $1$ no matter which state occurs
What is the price $p_c$ of the $S_c$ slip in the market for slips (before resolution of uncertainty)?
   It must equal $1$ if there is no significant time delay between buying/selling
   these contracts and the resolving of uncertainty
(If there is delay, then $p_c = 1/(1+r)$ where $r$ is the riskless rate of interest; ignore for now.)

Must have $p_1 + p_2 = p_c = 1$, regardless of whether there are any risk-neutral traders
Argument: [1] If $p_1 + p_2 > p_c$, simultaneously buy one $S_c$ and sell 1 each of $S_1$, $S_2$
   Net profit $p_1 + p_2 - p_c > 0$ earned right now and riskless
   After uncertainty resolves, collect $1$ on the $S_c$, to pay off $1$ on $S_1$ or $S_2$ depending on state
   As people compete to exploit this opportunity, they will bid down $p_1$, $p_2$
   [2] If $p_1 + p_2 < p_c$, simultaneously sell one $S_c$ and buy 1 each of $S_1$, $S_2$
   Net profit $p_c - p_1 - p_2 > 0$ earned right now and riskless
   After uncertainty resolves, collect $1$ on $S_1$ or $S_2$ depending on state, and pay off $1$ on $S_c$
   As people compete to exploit this opportunity, they will bid up $p_1$, $p_2$

Arbitrage: purchasing a set of financial assets at a low price and selling an equivalent or
repackaged set at a high price simultaneously. Arbitrageurs require no outlay of personal
endowment; revenue generated from the selling contract pays off the costs of the buying contract
and leaves a positive riskless net profit.

No-arbitrage principle: arbitrage opportunities cannot persist in equilibrium. This provides the
basic method for establishing relationships among prices of different financial assets.
TRADE IN CONTINGENT CLAIMS WHEN BOTH SIDES ARE RISK-AVERSE

EXAMPLE 1 – NO AGGREGATE RISK

Total quantities of contingent claims
(on wealth, income, consumption as relevant)
are equal in the two states – box is square
Total $W_0 = W_G + W_B$ (G: good, B: bad)
Two people, SW, NE. Their risks are
perfectly negatively correlated
Initial endowments are
SW: $(W_G, W_B)$, NE: $(W_B, W_G)$

SW’s budget constraint is
\[ p_1 W_1(SW) + p_2 W_2(SW) = p_1 W_G + p_2 W_B \]
He maxes
\[ EU = (1 - \pi) U_{SW}(W_1(SW)) + \pi U_{SW}(W_2(SW)) \]
If prices are statistically fair: $p_1 = 1 - \pi$, $p_2 = \pi$
he will choose full insurance, demands: $W_1(SW) = W_2(SW) = (1 - \pi) W_G + \pi W_B$

Similarly, $W_1(NE) = W_2(NE) = (1 - \pi) W_B + \pi W_G$
Then, in state 1, total contingent claims $W_1(SW) + W_1(NE) = W_G + W_B = W_0$
Similarly in state 2. So fair prices yield competitive general equilibrium
Both traders are fully insured: each has the same wealth in the two states
but SW has more wealth in both states than does NE if $\pi < \frac{1}{2}$ ;
conversely NE does better than SW if $\pi > \frac{1}{2}$
EXAMPLE 2 – AGGREGATE RISK

Total endowment $W_1 > W_2$ : state 1 is “good” and state 2 is “bad”

SW is less risk-averse than NE
(ICs less sharply curved)
So equilibrium is closer to NE’s 45-deg line than to SW’s

At any efficient risk-allocation,
$p_1 / p_2 < (1-\pi)/\pi$
So $p_2 > \pi$ and $p_1 < (1-\pi)$
and $p_2 - \pi = (1-\pi) - p_1$
Costs more now to buy claim to $1$ in bad state than probability,
because both are risk-averse and would want to buy at fair price

Today’s value of whole market
$= p_1 W_1 + p_2 W_2$
$= (1 - \pi) W_1 + \pi W_2 - (p_2 - \pi) (W_1 - W_2)$
$< (1 - \pi) W_1 + \pi W_2$
So buying whole market today yields excess expected return
This is aggregate risk premium; general equilibrium version of
the “price of market risk” of the mean-variance analysis in Handout 6 p. 10

INI = initial endowment, AB = core, C = equilibrium
Locus of Pareto efficient allocations
Equilibrium, $MRS = p_1 / p_2$
Points on 45-degree lines, $MRS = (1-\pi)/\pi$
SEcurities, Complete Markets, Spanning

A contingent claim to $1 in one state and nothing in any other state
is called an Arrow-Debreu security (ADS)

If there exist markets in Arrow-Debreu securities for all states, then
you can trade your initial ownership of contingent claims (ADSs), to obtain (consume)
any other point in contingent claims space subject only to the budget constraint

More typically, objects traded are not pure ADSs, but securities
Each security is a specific combination of contingent claims

If there are enough of these, then ADSs for all states of the world
can be constructed as linear combinations of other available securities

Example to show when and how this can be done:
Two states of the world: 1 - oil price is high, 2 - oil price is low
Securities: share ownership in two firms, A - oil company, B - auto company
Value (dividend etc) of each share: A: $2 in state 1, $1 in state 2. B: $1 in state 1, $3 in state 2

Suppose you want a pure state-1 ADS. Try holding x of firm-A shares and y of firm-B shares

Need 2x + 1y = 1; 1x + 3y = 0. Solution: x = 0.6, y = -0.2

Exercise: similarly find the combination that recreates a pure state-2 ADS.

Corresponding pricing relations:
Suppose shares in the two firms have prices \( \pi_A \), \( \pi_B \) respectively
What will be the prices \( P_1 \), \( P_2 \) of the ADSs?

No-arbitrage conditions: \( \pi_A = 2P_1 + 1P_2 \), \( \pi_B = 1P_1 + 3P_2 \)

Solving, \( P_1 = 0.6\pi_A - 0.2\pi_B \); exercise: find similar expression for \( P_2 \)
GENERAL THEORY

States of the world: \( s = 1, 2, \ldots, S \)

Prices (explicit or implicit) of pure Arrow-Debreu securities \( P_s \)

Firms' securities actually traded in markets: \( f = 1, 2, \ldots, F \)

- Firm \( f \)'s security yields \( a_{fs} \) in state \( s \)

Can we construct pure ADSs for each state as linear combinations of actually traded securities?

Do there exist \( X_{sf} \) such that,

\[
\sum_{f=1}^{F} X_{sf} a_{fs'} = \begin{cases} 
1 & \text{if } s' = s \\
0 & \text{if } s' \neq s 
\end{cases}
\]

(Negative \( X \)s are OK; they correspond to short sales.)

Answer: if the matrix \( (a_{fs}) \) has rank \( S \)

- i.e. the traded securities’ payoff vectors that span the state space

Then we say that there is a complete set of financial markets
Prices of firms’ securities $\Pi_f$ relate to prices $P_s$ of ADSs by the no-arbitrage conditions of market equilibrium:

$$\Pi_f = \sum_{s=1}^{S} a_{fs} P_s$$

So once we can price pure ADSs, we can also price any new security with any given payoff pattern across states of world

Examples: options and other derivative securities

Vector of prices of pure ADSs is “pricing kernel”

Conversely: given $\Pi_f$ determined in financial markets, will these equations determine $P_s$ uniquely?

If so, they become implicit prices of Arrow-Debreu securities even if such pure securities are not actually traded.

Answer: again, if the matrix $(a_{fs})$ has rank $S$

i.e. the payoff vectors of traded securities to span the state space

If $F > S$, can use submatrix of rank $S$ to create ADSs and then use no-arbitrage condition to price remaining $(F - S)$

Finance = General Equilibrium + Linear Algebra
Four-Scenario Example

Two farmers. Cora has Constant (relative) Risk Aversion:

\[ U(C) = \frac{1}{1 - \rho} C^{1 - \rho} \]

Ira has Infinite Risk Aversion. Output of each farmer can be either 1 or 2 with equal probability; independent. Four “states” with probability \( \frac{1}{4} \) each:
- \( g \) – “good state” – each has output 2; total output 4.
- \( b \) – “bad state” – each has 1; total 2.
- \( c \) – Cora has 2 and Ira hast 1; total 3.
- \( i \) – Cora has 1 and Ira has 2; total 3.

Cora’s budget constraint:

\[ P_g C_g^c + P_c C_c^c + P_i C_i^c + P_b C_b^c = 2 P_g + 2 P_c + P_i + P_b \]

Ira’s budget constraint is

\[ P_g C_g^i + P_c C_c^i + P_i C_i^i + P_b C_b^i = 2 P_g + P_c + 2 P_i + P_b \]

Equilibrium conditions: total demands must equal the total outputs in each state:

\[ C_g^c + C_i^i = 4, \ C_c^c + C_i^i = 3, \ C_i^c + C_i^i = 3, \ C_b^c + C_b^i = 2 \]

We can find three relative prices using any three of these equations. Numerical solution:

<table>
<thead>
<tr>
<th>Cora’s Risk-Aversion Coefficient ( \rho )</th>
<th>Cora’s Consumption Quantities in Scenarios</th>
<th>Ira’s Consumption Quantities (all Scenarios)</th>
<th>Prices of Arrow-Debreu Securities in Scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>g  2.50  c  1.50  i  1.50  b  0.50</td>
<td>g  1.50</td>
<td>g  0.99  c  1.00  i  1.00  b  1.01</td>
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<tr>
<td>0.50</td>
<td>g  2.60  c  1.60  i  1.60  b  0.60</td>
<td>g  1.40</td>
<td>g  0.78  c  1.00  i  1.00  b  1.64</td>
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<tr>
<td>1.00</td>
<td>g  2.68  c  1.68  i  1.68  b  0.68</td>
<td>g  1.32</td>
<td>g  0.63  c  1.00  i  1.00  b  2.44</td>
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<tr>
<td>2.00</td>
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<td>g  1.19</td>
<td>g  0.51  c  1.00  i  1.00  b  4.99</td>
</tr>
<tr>
<td>10.00</td>
<td>g  2.99  c  1.99  i  1.99  b  0.99</td>
<td>g  1.01</td>
<td>g  0.02  c  1.00  i  1.00  b  1013</td>
</tr>
</tbody>
</table>