13. Markets and Efficient Risk-Bearing: Examples and Extensions

1. Allocation of Risk in Mean-Variance Framework

S states of the world, probabilities \( \pi_s \) for \( s = 1, 2, \ldots S \)

One physical good ("wealth" or "corn"); aggregate quantities

\[
Y_s = \bar{Y} + y_s \quad \text{for } s = 1, 2, \ldots S
\]

with

\[
\bar{Y} = \sum_{s=1}^{S} \pi_s Y_s, \quad \sum_{s=1}^{S} \pi_s y_s = 0.
\]

Two people, A and B. Allocation

A gets \( X_s = \bar{X} + x_s \), \quad B gets \( Y_s - X_s = (Y - \bar{X}) + (y_s - x_s) \)

where

\[
\sum_{s=1}^{S} \pi_s x_s = 0, \quad \sum_{s=1}^{S} \pi_s (y_s - x_s) = 0
\]
Mean-variance objectives

\[ MV_A = X - \frac{1}{2} \alpha_A \sum_{s=1}^{S} \pi_s (x_s)^2, \quad MV_B = (Y - X) - \frac{1}{2} \alpha_B \sum_{s=1}^{S} \pi_s (y_s - x_s)^2. \]

Efficient allocation: choose \( X, (x_s) \) to \( \max MV_A \) subject to \( MV_B \geq k \).

Varying \( k \) over its range will trace out the Pareto frontier.

Lagrangian (with multipliers \( \lambda, \mu \)):

\[ L = \left[ X - \frac{1}{2} \alpha_A \sum_{s=1}^{S} \pi_s (x_s)^2 \right] + \lambda \left[ (Y - X) - \frac{1}{2} \alpha_B \sum_{s=1}^{S} \pi_s (y_s - x_s)^2 - k \right] + \mu \sum_{s=1}^{S} \pi_s x_s \]

For non-end-point maximization over \( X \),

\[ \frac{\partial L}{\partial X} = 1 - \lambda = 0. \]

Then

\[ L = Y - \frac{1}{2} \alpha_A \sum_{s=1}^{S} \pi_s (x_s)^2 - \frac{1}{2} \alpha_B \sum_{s=1}^{S} \pi_s (y_s - x_s)^2 - k + \mu \sum_{s=1}^{S} \pi_s x_s. \]

So varying \( k \) varies \( X \) over its range and traces Pareto frontier.
With respect to each $x_\sigma$,

$$\frac{\partial \mathcal{L}}{\partial x_\sigma} = -\alpha_A \pi_\sigma x_\sigma + \alpha_B \pi_\sigma (y_\sigma - x_\sigma) + \mu \pi_\sigma = 0$$

Sum over $\sigma$ and use conditions on sums of deviations and probabilities

$$-\alpha_A \ast 0 + \alpha_B \ast 0 + \mu \ast 1 = 0$$

or $\mu = 0$.

Then

$$-\alpha_A \pi_\sigma x_\sigma + \alpha_B \pi_\sigma (y_\sigma - x_\sigma) = 0$$

Yielding the solution

$$x_\sigma = \frac{\alpha_B}{\alpha_A + \alpha_b} y_\sigma, \quad y_\sigma - x_\sigma = \frac{\alpha_A}{\alpha_A + \alpha_b} y_\sigma$$

So each bears risk in inverse proportion to his/her coefficient of risk aversion.

When moral hazard is introduced, this will be modified for incentive reasons.
2. Incomplete Markets – Example

One physical good, corn. There are two farmers, A and B. Each is subject to risk.

A: Output, 30 or 10, probabilities \( \frac{1}{2} \) each (mean \( \mu_A = 20 \), std. dev. \( \sigma_A = 10 \))

B: Output 40 or 0, probabilities \( \frac{1}{2} \) each (mean \( \mu_B = 20 \), std. dev. \( \sigma_B = 20 \))

Independent risks. Four states of the world, probabilities \( \frac{1}{4} \) each.

Aggregate outputs (A’s label first) HH: 70; LH: 50; HL: 30; LL: 10

Mean aggregate output \( Y = 40 \), deviations \( y_s \) = respectively 30, 10, -10 and -30.

Mean-variance objective (utility) functions

\[
MV_A = \mu_A - \frac{1}{2} \sigma_A^2 \\
MV_B = \mu_B - \frac{1}{20} \sigma_B^2
\]

Constant absolute risk aversion coefficients \( \alpha_A = \frac{2}{5} \), \( \alpha_B = \frac{1}{10} \).

Without any trade in risk, utility levels

\[
MV_A = 20 - \frac{1}{5} 100 = 0 \\
MV_B = 20 - \frac{1}{20} 400 = 0.
\]

Think of the zeroes as choice of origin of utility.
**Pareto Efficient Allocation:**

A is four times as risk-averse as B, so the deviations should go

\[
\text{A: } 6, 2, -2, -6; \quad \text{B: } 24, 8, -8, -24.
\]

Resulting variances: A: 20, B: 320.

Utilities: \( MV_A = X - 4, \ MV_B = 40 - X - 16 = 24 - X \).

So any \( X \) in the interval (4, 24) is Pareto superior to no trade.

Pareto frontier: \( MV_A + MV_B = 20 \).

May have further restrictions to keep quantities non-negative in each state.

**Allocation Using Shares:**

B gets fraction \( \phi \) of A’s farm, A gets fraction \( (1 - \psi) \) of B’s farm.

Table of final consumption quantities:

<table>
<thead>
<tr>
<th>State</th>
<th>HH</th>
<th>LH</th>
<th>HL</th>
<th>LL</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>30(1 - ( \phi )) + 40(1 - ( \psi ))</td>
<td>10(1 - ( \phi )) + 40(1 - ( \psi ))</td>
<td>30(1 - ( \phi ))</td>
<td>10(1 - ( \phi ))</td>
</tr>
<tr>
<td>B</td>
<td>30( \phi ) + 40 ( \psi )</td>
<td>10 ( \phi ) + 40 ( \psi )</td>
<td>30 ( \phi )</td>
<td>10 ( \phi )</td>
</tr>
</tbody>
</table>
Varying $(\phi, \psi)$ over a large grid of values in the unit square produces feasible set and frontier using shares to reallocate risk, and comparison with full Pareto efficient frontier. Full efficiency is possible only exceptionally, at $(4, 16)$, with $X = 8$.

Figure 1: Frontiers With and Without Complete Markets
Implementing a point on the full-efficiency frontier using only shares is possible in exceptional cases at most. Here we see that the two frontiers do have one point in common, namely $(4, 16)$. To achieve this point, we need $X = 8$, and then A gets 14, 10, 6, 2 whereas B gets 56, 50, 24, 8. This is done by giving A a 20% share and giving B 80% share in each person’s farm. But that is a special feature of the mean-variance setup. With more general utility functions, efficient allocation may require a pattern of consumption across states of the world that cannot be reproduced using shares (that is, patterns of production across states of nature). Then the share-constrained efficient frontier may not have any fully efficient points at all.

3. Incomplete Markets – Some General Theory
The following material is algebraically somewhat heavy. We will go through it in class for the broad ideas, but you are not expected to be able to reproduce the detailed calculations. With complete markets, we have the usual equivalence between Pareto efficiency and competitive equilibria: a competitive equilibrium is Pareto efficient, and any Pareto efficient allocation can be achieved as a competitive equilibrium starting from suitably chosen initial endowments (or lump sums of purchasing power). Is there a similar correspondence between efficiency and equilibrium with incomplete markets? Does a competitive equilibrium in a limited set of markets for contingent claims have a property of constrained Pareto efficiency, in the sense that a government that must reallocate wealth in different state of the world obeying the same set of restrictions as the ones on market contracts cannot make one person better off without making someone else worse off? Here we will find that to be the case in a simple model of trading in shares. In more general contexts the equivalence breaks down; that must be left for more advanced courses.

Suppose there are two people (allowing more people just complicates the algebra without...
3. Incomplete Markets – Constrained Efficiency?

Can have a market where A and B trade shares in their enterprises.
Can social planner improve on such a competitive general equilibrium
using only shares (not full AD securities) for reallocation purposes?

Two people $A, B$. States $s = 1, 2, \ldots S$.
Endowments $(X^0_{i1}, X^0_{i2}, \ldots X^0_{iS})$ for $i = A, B$.
Final consumption quantities $(X_{i1}, X_{i2}, \ldots X_{iS})$.
Feasible allocation: $X_{As} + X_{Bs} \leq X^0_{As} + X^0_{Bs}$ for all $s$.
Utilities $U_i(X_{i1}, X_{i2}, \ldots X_{iS})$ for $i = A, B$.

Ideal full Pareto efficiency for comparison:
All the $X_{is}$ are independent choice variables.
Usual Pareto efficiency conditions. For any two states $s, t$:

$$\frac{\partial U_A}{\partial X_{As}} = \frac{\partial U_B}{\partial X_{Bs}} \Rightarrow \frac{\partial U_A}{\partial X_{At}} = \frac{\partial U_B}{\partial X_{Bt}}$$

Therefore for all states $s$,

$$\frac{\partial U_A}{\partial X_{As}} = \theta.$$
Using shares only, reallocations are restricted to two degrees of freedom:

\[ X_{As} = (1 - \phi) X^0_{As} + (1 - \psi) X^0_{Bs}, \quad X_{Bs} = \phi X^0_{As} + \psi X^0_{Bs}. \]

So efficiency condition

\[ \frac{\partial U_A / \partial \phi}{\partial U_A / \partial \psi} = \frac{\partial U_B / \partial \phi}{\partial U_B / \partial \psi} \]

or

\[ \sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}} X^0_{As} = \sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}} X^0_{Bs}, \quad \text{or} \quad \sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}} X^0_{As} = \sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}} X^0_{Bs} = \nu \]

Full efficiency implies this condition (with \( \nu = \theta \)), but not conversely except in very special cases (\( S = 2 \) and output patterns not perfectly correlated).
In market, let $p = \text{price of } A\text{'s enterprise relative to } B\text{'s.}

$A$ sells fraction $\phi$ of his enterprise to get $p\phi$ of $B$'s. So

$$X_{As} = (1 - \phi) X_{As}^0 + p\phi X_{Bs}^0.$$ 

Condition for optimal choice of $\phi$:

$$\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}} [-X_{As}^0 + p X_{Bs}^0] = 0,$$

or

$$\frac{\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}} X_{As}^0}{\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}} X_{Bs}^0} = p.$$

Similarly for $B$. Therefore the constrained efficiency condition is met.

However, this result does not generalize to many periods etc.