APPLICATIONS OF EXPENDITURE FUNCTION
1. EXACT COMPENSATION FOR PRICE CHANGES

Compare two situations: Old A:
prices \((P^a_x, P^a_y)\), quantities \((x^a, y^a)\), utility \(u^a\)

New B: Change \(P_x\) only, to \(P^b_x\)

Could get old utility by consuming old quantities:
That would cost \(P^b_x x^a + P^a_y y^a\)
New expenditure-minimizing bundle cannot cost more:
\[ M^*(P^b_x, P^a_y, u^a) \leq P^b_x x^a + P^a_y y^a \]
Graphing each of the two sides against $P_x$, the line is tangent to graph of $M^*$ at A.
The graph of $M^*$ lies everywhere below the tangent. Its slope at A is
\[ x^a = \left. \frac{\partial M^*}{\partial P_x} \right|_{\text{at } a} \]
Since A could be any point, this proves concavity and Hotelling’s Lemma in one step.
If no substitution: $(x^a, y^a)$ only way to achieve $u^a$, $M^*(P_x, P_y, u^a) = P_x x^a + P_y y^a$ linear in prices coincides with tangent.
Thus possibility of substitution of quantities makes expenditure function concave.
Substitution occurs when relative prices change.
If all prices change in same proportion as with pure inflation then $(P_x^b, P_y^b) = k \cdot (P_x^a, P_y^a)$, no subst’n and $M^*(P_x^b, P_y^b, u^a) = k \cdot M^*(P_x^a, P_y^a, u^a)$.
Initial budget line A, optimal choice X
After increase in price of $x$,
If no compensation, budget line B.
If (Hicks) compensation to maintain old utility:
   Budget line BH, choice H
If (Slutsky) compensation to allow purchase of old X:
   Budget line BS

Slutsky compensation $= (P^b_x - P^a_x) x^a$ (too much:
   consumer “cheats,” consumes S for higher utility)
Hicks comp’$n = M^*(P^b_x, P^a_y, u^a) - M^*(P^a_x, P^a_y, u^a)$

H, S coincide with X if L-shaped indiff. curves
2. COST OF LIVING INDEXES

Initial point $A$: prices $P^a$, income $I^a$
quantities $x^a$, utility $u^a = U^*(P^a, I^a)$

New prices $P^b$. To preserve old utility, need income
$M^*(P^b, u^a) = M^*(P^b, U^*(P^a, I^a))$

Note: combination of $M^*$ and $U^*$ cancels effect of
special cardinal choice of utility ("anchoring")

True cost of living index: $M^*(P^b, u^a) / P^a \cdot x^a$

Usual initial quantity-based index: $P^b \cdot x^a / P^a \cdot x^a$

True index $\leq$ Conventional index, with equality
only if no substitution or all prices change in proportion

Examples: (1) Bias in the consumer price index over time
Relative prices change (teleconferencing vs. meeting)
(2) *The Economist*'s cost of living index for cities:
3. DEAD-WEIGHT BURDEN OF TAX ON GOODS

Lump-sum tax $T$: Given $M$, $P_x$, $P_y$,
consumer achieves $u$ defined by $M-T = M^*(P_x, P_y, u)$
OR: Tax $t$ per unit of good $x$ keeping utility same:

$$M = M^*(P_x + t, P_y, u), \quad x = \frac{\partial M^*}{\partial P_x} \bigg|_{(P_x+t,P_y,u)}$$

Revenue raised by this tax: $R = tx$.
How do the two compare? Take Taylor expansion of $M^*$
around the “base point” $(P_x + t, P_y, u)$:

$$M^*(P_x, P_y, u) \approx M^*(P_x+t, P_y, u) + (-t) \frac{\partial M^*}{\partial P_x} + \frac{1}{2} (-t)^2 \frac{\partial^2 M^*}{\partial P_x^2}$$

Write

$$\Delta x = -t \frac{\partial x}{\partial P_x} \bigg|_{u = \text{constant}} = -t \frac{\partial^2 M^*}{\partial P_x^2},$$

pure substitution part of the reduction in $x$ due to tax.
Then

$$M - T \approx M - tx - \frac{1}{2} t \Delta x$$

$$R \approx T - \frac{1}{2} t \Delta x.$$ 

So lump-sum tax better. Reason: substitution.
Difference is the dead-weight loss.
This fits with the usual “consumer surplus” idea: Tax raises the price from OA to OD, and quantity falls from AC to DE = AB. Tax revenue = rectangle ABED. Dead-weight loss or Excess burden of tax = triangle BCE. These sum to trapezoid ACED = Loss of consumer surplus.

New feature, different from ECO 102: Consumer surplus better measured as the area to the left of the Hicksian demand curve, not the Marshallian demand curve.