GENERAL EQUILIBRIUM AND EFFICIENCY
– MATHEMATICAL VERSION

Assumptions
Competitive (price-taking) behavior;
this needs concave production functions
No information asymmetries (for people or planner)
Complete markets. No externalities or public goods

Notation:
Individuals (households) \( h = 1, 2 \ldots H \)
Goods \( g = 1, 2 \ldots G \). Factors \( f = 1, 2 \ldots F \).

\[
\begin{align*}
X_h &= (x_{hg}) = \text{consumption quantities of } h \\
V_h &= (v_{hf}) = \text{factors supplied by } h \\
Z_g &= (z_{fg}) = \text{factor used by firm / good } g \\
U_h(X_h, V_h) &= \text{utility function of } h \\
X_g(Z_g) &= \text{production function of firm / good } g
\end{align*}
\]

Constraints: Material balance for goods \( g \), factors \( f \):
\[
\begin{align*}
x_{1g} + x_{2g} + \ldots + x_{Hg} & \leq X_g \\
z_{f1} + z_{f2} + \ldots + z_{fG} & \leq v_{1f} + v_{2f} + \ldots + v_{Hf}
\end{align*}
\]
Optimization

Social objective (Pareto efficiency + interpersonal weights)

\[ W = W(U_1, U_2 \ldots U_H) \]

\[ \mathcal{L} = W + \sum_{g=1}^{G} \alpha_g \left[ X_g - \sum_{h=1}^{H} x_{hg} \right] + \sum_{f=1}^{F} \beta_f \left[ \sum_{h=1}^{H} v_{hf} - \sum_{g=1}^{G} z_{fg} \right] \]

FONCs for interior optimum:

\[ \frac{\partial W}{\partial U_h} \frac{\partial U_h}{\partial x_{hg}} - \alpha_g = 0 \]
\[ \frac{\partial W}{\partial U_h} \frac{\partial U_h}{\partial v_{hf}} + \beta_f = 0 \]
\[ \alpha_g \frac{\partial X_g}{\partial z_{fg}} - \beta_f = 0 \]

Relate to Edgeworth Box geometry of Nov. 4 handout:

Exchange (Fig. 1) \[ \frac{\partial U_h}{\partial x_{h1}} / \partial U_h / \partial x_{h2} = \frac{\alpha_1}{\alpha_2} \] for all \( h \)

Factor alloc (Fig. 3) \[ \frac{\partial X_g}{\partial z_{1g}} / \partial X_g / \partial z_{2g} = \frac{\beta_1}{\beta_2} \] for all \( g \)

Combined (Fig. 4) \[ -\frac{\partial U_h}{\partial v_{hf}} / \partial U_h / \partial x_{hg} = \frac{\beta_f}{\alpha_g} = \frac{\partial X_g}{\partial z_{fg}} \] for all \( h \)
Equilibrium

\( p = (p_g), \ w = (w_f) \) vectors of prices of goods factors

**FIRMS**

Profit function \( \pi_g(p_g, w) = \max p_g X_g(z_{1g}, \ldots z_{Fg}) - \sum_{f=1}^{F} w_f z_{fg} \)

\[ X_g = \frac{\partial \pi_g}{\partial p_g}, \quad z_{fg} = -\frac{\partial \pi_g}{\partial w_f} \]

**HOUSEHOLDS**

\( p \cdot X_h \leq w \cdot V_h + I_h \)

\( x_{hg} = x_{hg}(p, w, I_h) \), \quad \nu_{hf} = \nu_{hf}(p, w, I_h) \)

\( I_h = \sum_{g=1}^{G} \theta_{hg} \pi_g(p_g, w) \)

**EQUILIBRIUM**

\[
\sum_{h=1}^{H} x_{hg} \left( p, w, \sum_{g=1}^{G} \theta_{hg} \pi_g(p_g, w) \right) = X_g(p_g, w)
\]

\[
\sum_{g=1}^{G} z_{fg}(p_g, w) = \sum_{h=1}^{H} \nu_{hf} \left( p, w, \sum_{g=1}^{G} \theta_{hg} \pi_g(p_g, w) \right)
\]
Existence

System has \((F + G - 1)\) unknowns and equations:

1. All quantities homogeneous degree zero in \((p, w)\)
   - Only relative prices matter
   - Can use freedom to set any one \(p_g\) or \(w_f\) equal to 1
   - Then all prices measured in units of this good or factor
   - It is called *numeraire* – can be composite bundle

2. Walras’ Law - at all \((p, w)\) (equil. or not)
   - Total value of excess demands is \(\equiv 0\)

\[
\sum_{g=1}^{G} p_g \left[ \sum_{h=1}^{H} x_{hg} - X_g \right] + \sum_{f=1}^{F} w_f \left[ \sum_{g=1}^{G} z_{fg} - \sum_{h=1}^{H} v_{hf} \right] = \sum_{h=1}^{H} \left[ \sum_{g=1}^{G} p_g x_{hg} - \sum_{f=1}^{F} w_f v_{hf} \right] - \sum_{g=1}^{G} \left[ p_g X_g - \sum_{f=1}^{F} w_f z_{fg} \right] = \sum_{h=1}^{H} I_h - \sum_{g=1}^{G} \pi_g \equiv 0
\]

Existence of solution proved by fixed point theorem
Uniqueness, dynamic stability not guaranteed
Equivalence

In equilibrium, consumers’ and firms’ FONCs are

\[
\frac{\partial U_h}{\partial x_{hg}} - \lambda_h p_g = 0, \quad \frac{\partial U_h}{\partial v_{hf}} + \lambda_h w_f = 0
\]

\[
p_g \frac{\partial X_g}{\partial z_{fg}} = w_f
\]

Same as Lagrange conditions of social optimum if

\[
\frac{\partial W}{\partial U_h} = 1/\lambda_h
\]

with

\[
\alpha_g = p_g, \quad \beta_f = w_f
\]

So equilibrium is an optimum with particular social weights
It is “Pareto efficient”

Conversely, social optimum with planner’s weights can be implemented as equilibrium if lump sums can be transferred between people to make

\[
\lambda_h = 1/(\partial W/\partial U_h)
\]
Uses of general equilibrium analysis

(1) Emphasizes linkages throughout economy
(2) Clarifies conditions needed for optimality of markets
   perfect competition, no externalities or public goods
   complete markets, no information asymmetries
   No guarantee of ethically good distribution of income
(3) Extensions to allow externalities, public goods
   can find optimal policy intervention
(4) Clarifies limitations of planning
   benevolent dictator with complete information
(5) Broad interpretation
   (a) equilibrium over time: $p_{t-1}/p_t = 1 + r$
      borrowing and lending, investment, R-and-D
   (b) uncertainty: claims to $ in different “scenarios”
      trading risk, insurance, finance

Other extensions/modifications more drastic
   (a) Oligopoly – strategic behavior, game theory
   (b) Asymmetric information – signaling, screening
      incentive schemes, contracts and organizations