PORTFOLIO CHOICE

One Riskless, One Risky Asset

Safe asset: gross return rate $R$ (1 plus interest rate)
Risky asset: random gross return rate $r$

Mean $\mu = \mathbb{E}[r] > R$, Variance $\sigma^2 = \mathbb{V}[r]$

Initial wealth $W_0$. If $x$ in risky asset,

final wealth $W = (W_0 - x)R + x\mu = R W_0 + (\mu - R) x$

$\mathbb{E}[W] = W_0 R + x(\mu - R)$
$\mathbb{V}[W] = x^2 \sigma^2$; Std. Dev. $= x \sigma$

As $x$ varies, straight line in (Mean, Std.Dev.) figure.

$P_s = (0, W_0 R)$ safe; $P_r = (W_0 \sigma, W_0 \mu)$ risky;

Beyond $P_r$ possible if leveraged borrowing OK
Objective function Mean $- a (\text{Std.Dev.})^2$; so $P^*$ optimal
Two Risky Assets

\[ W_0 = 1; \] Random gross return rates \( r_1, r_2 \)
Means \( \mu_1 > \mu_2; \) Std. Devs. \( \sigma_1, \sigma_2, \) Correl. Coefft. \( \rho \)
Portfolio \((x, 1-x)\). Final \( W = x r_1 + (1-x) r_2 \)

\[ E[W] = x \mu_1 + (1-x) \mu_2 = \mu_2 + x (\mu_1 - \mu_2) \]

\[ V[W] = x^2 (\sigma_1)^2 + (1-x)^2 (\sigma_2)^2 + 2 x (1-x) \rho \sigma_1 \sigma_2 \]
\[ = (\sigma_2)^2 - 2 x \sigma_2 (\sigma_2 - \rho \sigma_1) + x^2 [(\sigma_1)^2 - 2 \rho \sigma_1 \sigma_2 + (\sigma_2)^2] \]

Diversification can reduce variance if \( \rho < \min [\sigma_1/\sigma_2, \sigma_2/\sigma_1] \)
\( P_1, P_2 \) points for the two individual assets
\( P_m \) minimum-variance portfolio
Portion \( P_2 P_m \) dominated; \( P_m P_1 \) efficient frontier
Continuation past \( P_1 \) if short sales of 2 OK
Optimum \( P^* \) when preferences as shown
One Riskless, Two Risky Assets

First combine two riskies; then mix with riskless

This gets all points like $P_h$ on all lines like $P_s P_r$
Efficient frontier $P_s P_F$ tangential to risky combination curve
Then along curve segment $P_F P_1$ if no leveraged borrowing;
continue straight line $P_s P_F$ if leveraged borrowing OK

With preferences as shown, optimum $P^*$
mixes safe asset with particular risky combination $P_F$
“Mutual fund” $P_F$ is the same for all investors
regardless of risk-aversion (so long as optimum in $P_s P_F$)

Even less risk-averse people may go beyond $P_F$
including corner solution at $P_1$
or tangency past $P_1$ if can sell 2 short to buy more 1
CAPITAL ASSET PRICING MODEL

Individual investors take the rates of return as given but these must be determined in equilibrium. Add supply side – firms issue equities.

Take production, profit-max as exogenous.

Two firms, profits $\Pi_1$ and $\Pi_2$. Means $E[\Pi_1], E[\Pi_2]$; Variances $V[\Pi_1], V[\Pi_2]$; Covariance $Cov[\Pi_1, \Pi_2]$.

Safe asset (government bond) sure gross return rate $R$.

Market values of firms $F_1, F_2$; to be solved for (endogenous) (Random) rates of return $r_1 = \Pi_1/F_1$ and $r_2 = \Pi_2/F_2$.

and for whole market, $r_m = (\Pi_1 + \Pi_2)/(F_1 + F_2)$

After a lot of algebra, important results:

(1) $E[r_1] - R = \frac{Cov[r_1, r_m]}{V[r_m]} \{ E[r_m] - R \}$

Risk premium on firm-1 stock depends on its systematic risk (correlation with whole market) only, not idiosyncratic risk (part uncorrelated with market).

Coefficient is beta of firm-1 stock

(2) $F_1 = \frac{E[\Pi_1] - A \cdot Cov[\Pi_1, \Pi_1 + \Pi_2]}{R}$

where $A$ is the market’s aggregate risk-aversion (usually small).

Value of firm = present value of its profits adjusted for systematic risk, and discounted at riskless rate of interest.
ROCKET-SCIENCE FINANCE

Equity, debt etc - complex pattern of payoffs in different scenarios: vector \( S = (S_1, S_2, \ldots) \)

Owning security \( S \) is full equivalent to owning portfolio of Arrow-Debreu securities (ADS):
\( S_1 \) of \( ADS_1 \), \( S_2 \) of \( ADS_2 \), \ldots

In equilibrium, no “riskless arbitrage” profit available
So relation bet. price \( P_S \) of \( S \) and ADS prices \( p_i \):
\[
P_S = S_1 p_1 + S_2 p_2 + \ldots
\]

Converse example: Two scenarios, two firms’s shares payoff MicTel \((M_1, M_2)\), BioWiz \((B_1, B_2)\).
If \( X_M \) of MicTel + \( X_B \) of BioWiz \( \equiv 1 \) of \( ADS_1 \),
\[
X_M \ M_1 + X_B \ B_1 = 1, \quad X_M \ M_2 + X_B \ B_2 = 0
\]
\[
X_M = \frac{B_2}{M_1 B_2 - B_1 M_2}, \quad X_B = \frac{-M_2}{M_1 B_2 - B_1 M_2}
\]

One of these may be negative: need short sales
ADS’s can be “constructed” from available securities
Then no-arbitrage-in-equilibrium condition:

\[ P_1 = X_M P_{\text{MicTel}} + X_B P_{\text{BioWiz}} \]

Similarly \( P_2 \). So the “constructed” ADS’s can be priced.

Every financial asset is defined by its vector of payoffs in all scenarios. Therefore it can be priced using these prices of all ADS’s (“pricing kernel”)
Examples – options and other derivatives

General idea: Markets for risks are complete, and achieve Pareto-efficient allocation of risks if enough securities exist that their payoff vectors span the space of wealths in all scenarios

“Rocket-science finance” extends this idea to infinite-dimensional spaces of scenarios
If sequence of periods, need enough markets to span the scenarios one-period ahead, and then rebalance portfolio by trade (dynamic hedging)
Finnece = General equilibrium + Linear algebra!

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