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A NEW MEAN VELOCITY SCALING FOR TURBULENT BOUNDARY LAYERS

Mark V. Zagarola
Creare Incorporated
Etna Road • P.O. Box 71
Hanover, New Hampshire 03755
mvz@creare.com

Alexander J. Smits
Department of Mechanical & Aerospace Engineering
Princeton University
Princeton, New Jersey 08544
asmits@princeton.edu

ABSTRACT

A new scaling, originally developed for the mean velocity profile of turbulent pipe flow, was extended to the case of zero pressure gradient turbulent boundary layers. At low Reynolds numbers, the new scaling leads to a power law for the overlap region of the mean velocity profile. At high Reynolds numbers, the conventional overlap region given by a log law is also obtained. Comparisons were made between the power law and 17 velocity profiles covering a large range of Reynolds numbers. This comparison showed that a power law with empirical constants determined from pipe flow data was in good agreement with boundary layer data. A new outer velocity scale was also proposed. The new outer velocity scale was used to normalize the 17 velocity profiles and the collapse of these profiles was significantly better than for profiles normalized by the friction velocity or the freestream velocity.

KEYWORDS: Boundary layers, overlap region, pipe flow, mean-velocity profile.

NOMENCLATURE

B	Empirical constant in log law
C_1	Empirical constant in power law
C_f	Skin-friction coefficient = $2\,(u_\tau/U_\infty)^2$
δ	Boundary layer thickness
δ^*	Displacement thickness
δ^+	Ratio of outer to inner length scales for a boundary layer = $\delta u_\tau/\nu$
f	Function of y^+
γ	Empirical constant in power law
g	Function of η
η	Wall-normal distance normalized by outer length scale = y/R or y/δ
κ	Empirical constant in log law
Λ	Ratio of outer to inner velocity scales = u_o/u_τ
ν	Kinematic viscosity
Θ	Momentum thickness
ρ	Density
R	Pipe radius
R^+	Ratio of outer to inner length scales for a pipe = Ru_τ/ν
Re_Θ	Reynolds number based on momentum thickness = $U_\infty\Theta/\nu$
τ_w	Wall shear stress
U	Streamwise velocity
U^+	Velocity normalized by friction velocity = U/u_τ
U_{CL}	Centerline velocity
U_∞	Free-stream velocity
$\overline{U}$	Average velocity
u_o	Outer velocity scale
u_τ	Friction velocity = $\sqrt{\tau_w/\rho}$
x	Streamwise distance
y	Wall-normal distance
y^+	Wall-normal distance normalized by inner length scale = yu_τ/ν

BACKGROUND

In this paper a new scaling argument, originally developed for the mean velocity profile in turbulent pipe flow, is extended to the case of turbulent boundary layers. The original argument is based on the observations made in the Princeton University pipe flow experiment which covered over three orders of magnitude in Reynolds number (Zagarola & Smits, 1997). There it was shown that at sufficiently high Reynolds numbers the mean velocity profile in a pipe consists of two overlap regions. At small Reynolds numbers, a single overlap region exists, and the mean velocity profile in this region can be represented by a power law. The power law exists in a discrete region between the inner and outer region or between the inner and logarithmic overlap region, depending on the magnitude of the Reynolds number, and the empirical constants in the power law do not depend on Reynolds number when expressed using inner scaling variables. This region is not the overlap region expected at very large Reynolds number, but an intermediate overlap region that covers the range of y^+ at which most previous experiments have been performed. At very large Reynolds number, a second overlap region is apparent, and the scaling in this region was shown to be logarithmic.

An overlap argument was developed that is consistent with this behavior. A new velocity scale is required for the outer region such that the ratio of the outer velocity scale to the inner velocity scale (the friction velocity) is a function of Reynolds number at low Reynolds numbers, and approaches a constant value at high Reynolds numbers. A reasonable candidate for the outer velocity scale is the velocity deficit in the pipe. In this paper, we will review the arguments used to derive the two overlap regions for pipe flow, extend these arguments to boundary layers, and compare the new scaling for boundary layers with existing experimental data. The boundary layer analysis and comparisons will be confined here to the incompressible case with no streamwise pressure gradient.

For wall-bounded turbulent shear flows, the shape of the mean velocity profile, or equivalently, the relative fraction of the flow occupied by the inner and outer regions, changes with Reynolds number. If the Reynolds number is large enough, it is usually assumed that the interaction between these regions vanishes because of the disparity of length scales, and consequently, independent similarity solutions may exist for each region. Therefore, most theoretical treatments start by dividing the flow into an inner and outer region. For each region, a length and velocity scale may be defined. The velocity scale in the near-wall region is typically taken to be the friction velocity. The length scale associated with the inner region is then the kinematic viscosity ν divided by the friction velocity, ν/u_τ. For the outer region, the velocity scale is also typically taken to be the friction velocity, although this has long been the source of controversy (Zagarola & Smits, 1997; George et al., 1996), and the length scale is taken to be the radius of the pipe R or the boundary layer thickness δ.

Using dimensional analysis, the scaling for the inner region is

$$U^+ = f\left(y^+\right) \tag{1}$$

where f represents the functional dependence in the inner region (see Schlichting, 1987). Equation 2 is known as the “law-of-the-wall” and is valid only in the inner region. It can be shown from the Navier-Stokes equation that f is linear near the wall, and we may expect that Equation 1 is valid further from the wall than the linear region but not into the outer region (i.e. Equation 1 will hold for $0 < y^+ << R^+$).

The dimensionless scaling law for the outer region is

$$\frac{U_{CL} - U}{u_o} = g(\eta) \tag{2}$$

where g represents the functional dependence in the outer region and $\eta = y/R$ for a pipe. If $u_o = u_\tau$, then Equation 2 is known as the “defect-law” (see Schlichting, 1987). Equation 2 is valid only in the outer region where viscosity is not important (i.e., Equation 2 will hold for $0 << \eta < 1$).

Equations 1 and 2 are based on the assumption that R^+ is large enough for both regions to be independent of Reynolds number. If we assume that an intermediate region exists where both scaling laws are valid, then we can define two different matching conditions. By matching the velocity gradients given by Equations 1 and 2, we find

$$y^+ f' = -\Lambda \eta g' \tag{3}$$

where the differentiation in Equation 3 is with respect to the dependent variables and Λ is the ratio of the outer to inner velocity scales, u_o/u_τ. If $u_o = u_\tau$, then Equation 3 is the same relation used by Millikan (1938) to derive the classical logarithmic overlap region.

Alternatively, if we simultaneously match the velocities and velocity gradients, the matching condition is

$$y^+ \frac{f'}{f} = -\frac{\eta g'}{\frac{U_{CL}}{u_o} - g} \tag{4}$$

Equation 4 is the same relation used by George et al. (1996) with $u_o = U_\infty$ to support their assertion that the overlap region in a boundary layer is given by a power law.

We argue that at low Reynolds numbers, but still high enough that an overlap region exists, Λ depends on R^+. At these Reynolds numbers, Equation 3 does not define an overlap region that is independent of R^+, but Equation 4 does. By integrating Equation 4, the velocity profile in this region can be written using inner layer variables as

$$U^+ = C_1\left(y^+\right)^\gamma \tag{5}$$

For pipe flow, the values of C_1 and γ were shown to be independent of Reynolds number and equal to 8.70 and 0.137, respectively (Zagarola & Smits, 1997). Equation 5 with these constants was shown to be in excellent agreement with pipe flow data for $60 < y^+ < 500$ or $y^+ < 0.15R^+$, the outer limit depending on whether R^+ is greater or less than 9×10^3 (Zagarola & Smits, 1998). With these limits, a power law can exist only if $R^+ > 400$.

At even higher Reynolds numbers, it was shown that u_o/u_τ approaches a finite limit (Zagarola & Smits, 1997). For this case, Equation 3 also gives an overlap region which is independent of Reynolds number. Equation 3 can be set equal to a constant (typically $1/\kappa$) and integrated to give the classical log law which can be written in terms of inner scaling variables as

$$U^+ = \frac{1}{\kappa}\ln y^+ + B \qquad (6)$$

The values of κ and B were shown to be 0.436 and 6.15, and this log law was shown to be in excellent agreement with experimental pipe flow data for $600 < y^+ < 0.07R^+$ (Zagarola & Smits, 1998). With these limits, a log law can exist only if $R^+ > 9 \times 10^3$ which is a very large Reynolds number compared to most laboratory flows.

For the preceding argument to be valid, u_o must be proportional to u_τ at high Reynolds number. The correct velocity scale for the outer region was shown to be the velocity deficit in the pipe, or $U_{CL} - \overline{U}$, which is a true outer velocity scale, in contrast to the friction velocity which is a velocity scale associated with the inner region which is "impressed" on the outer region (Zagarola & Smits, 1997).

SCALING OF TURBULENT BOUNDARY LAYERS

The preceding analysis for pipe flow may also hold for boundary layers if the centerline velocity is replaced by the freestream velocity and the radius is replaced by the boundary layer thickness. Here we also assume that the streamwise dependence of the velocity profile is properly accounted for by our choice of length and velocity scales. An outer velocity scale equivalent to $U_{CL} - \overline{U}$ can be expressed using boundary layer parameters as follows.

$$u_o = U_\infty - \overline{U} = \frac{U_\infty}{\delta}\int_0^\infty 1 - \frac{U}{U_\infty} dy = U_\infty \frac{\delta^*}{\delta} \qquad (7)$$

This new outer velocity scale can be accurately determined from the velocity profiles, in contrast to the friction velocity u_τ which is not easily measured accurately in a boundary layer. At high Reynolds numbers, we can expect that $u_o \sim u_\tau$, or equivalently $\delta^*/\delta \sim \sqrt{C_f}$, for a logarithmic overlap region to exist.

Even though a similar scaling may exist for boundary layers and pipe flow, we can not expect the functional form of the velocity profiles in the outer region $g(\eta)$ to be the same since the equations of motion and the boundary conditions are different. This is true even in the infinite Reynolds number limit. Furthermore, any limit that depends on Reynolds number (R^+ or δ^+) may be different due to the differences in the outer region. These limits include the Reynolds number at which complete similarity exists in the outer and inner region, the outer limit of the power law or log law, and the Reynolds number at which the overlap regions appear. Conversely, the equations of motion and boundary conditions of the inner region are the same for both flows in the infinite Reynolds number limit, and we may therefore expect that the functional form of the velocity profiles in the inner region $f(y^+)$ are the same.

The remainder of this paper is devoted to a comparison between the new scaling laws and experimental boundary layer data.

COMPARISON WITH EXPERIMENTS

Data from three separate investigations were used for the comparison presented here. The data from Purtell et al. (1981) spanned the low Reynolds number range; the data from Smith (1994) spanned the moderate Reynolds number range; and the data from Fernholz et al. (1995) spanned the high Reynolds number range. A summary of the relevant boundary layer parameters is given in Table 1.

Purtell et al. used a hot wire to measure the velocity profiles and inferred u_τ from an assumed log law with $\kappa = 0.41$ and B = 5.0. For comparison, they also inferred u_τ from dU/dy and $d\Theta/dx$, and found that the agreement was ± 2.5 % for the higher Reynolds numbers and was somewhat poorer for the lower Reynolds numbers. Smith in his experiment used a flattened Pitot probe to measure the velocity profiles and a Preston probe to measure u_τ. He also inferred u_τ from an assumed log law with $\kappa = 0.41$ and B = 5.2. The agreement between the different methods used to determine u_τ was $\pm$ 1 %. Fernholz et al. used a hot wire to measure the velocity profiles and inferred u_τ from an empirical relation based on the measurement of Re_Θ. They also measured u_τ using a Preston probe and inferred u_τ from several other empirical relations. The agreement between the different methods used to determine u_τ was $\pm$ 4%.

Table 1. Boundary Layer Parameters

Profile #	Re_Θ	δ^+	Reference
1	470	220	Purtell et al. (1981)
2	500	220	Purtell et al. (1981)
3	700	290	Purtell et al. (1981)
4	1.0×10^3	390	Purtell et al. (1981)
5	1.3×10^3	470	Purtell et al. (1981)
6	1.8×10^3	650	Purtell et al. (1981)
7	2.8×10^3	970	Purtell et al. (1981)
8	3.5×10^3	1.2×10^3	Purtell et al. (1981)
9	4.1×10^3	1.4×10^3	Purtell et al. (1981)
10	4.6×10^3	1.5×10^3	Smith (1994)
11	5.0×10^3	1.6×10^3	Smith (1994)
12	5.1×10^3	1.7×10^3	Purtell et al. (1981)
13	5.4×10^3	1.7×10^3	Smith (1994)
14	5.9×10^3	1.8×10^3	Smith (1994)
15	6.9×10^3	2.1×10^3	Smith (1994)
16	7.7×10^3	2.3×10^3	Smith (1994)
17	9.1×10^3	2.7×10^3	Smith (1994)
18	10×10^3	3.1×10^3	Smith (1994)
19	12×10^3	3.5×10^3	Smith (1994)
20	13×10^3	4.0×10^3	Smith (1994)
21	21×10^3	6.9×10^3	Fernholz et al. (1995)
22	58×10^3	18×10^3	Fernholz et al. (1995)

In Figure 1, the velocity profiles measured by Purtell et al., Smith, and Fernholz et al. are shown normalized by inner layer variables. The data cover a Reynolds number range given by $650 < \delta^+ < 18 \times 10^3$ or $4.6 \times 10^3 < Re_\Theta < 58 \times 10^3$. The data at lower values of δ^+ (Profiles 1 to 5) are not shown since it is doubtful that a universal overlap region exists at these Reynolds numbers ($\delta^+ < 500$). The power law established from pipe flow data is also shown, as are the regions marking a $\pm$ 3% error in u_τ. For all profiles except at the highest Reynolds number, the data are nominally within $\pm$ 3% of the power law for some range of y^+ and deviate from the curve in the inner region where viscosity dominates and in the outer region where the inner scaling no longer holds. At the highest Reynolds number, the data near the wall deviates from the other profiles by more than 3%, but this perhaps can be attributed to an error in position since the five points nearest to the wall are all within 1 mm of the wall. The log law established from pipe flow data is also shown in Figure 1. According to our analysis of pipe flow data, the log law should be apparent only at the highest Reynolds number since a log law should not exist until δ^+ is of order 10^4. The uncertainty in the friction velocity prevents us from drawing any definitive conclusions here, but a power law with $C_1 = 8.70$ and $\gamma = 0.137$ seems to be in good agreement with these boundary layer data.

In Figures 2, 3 and 4, the velocity profiles are normalized by outer layer variables. The conventional outer velocity scale, u_τ, is used to normalize the profiles in Figure 2, the proposed outer velocity scale $U_\infty\delta^*/\delta$ is used in Figure 3, and the outer velocity scale proposed by George et al. (1996), U_∞, is used in Figure 4. For comparison between these figures, error bars are shown which represent a $\pm$ 3% uncertainty of the ordinate at $y/\delta = 0.1$. When normalizing the wall-normal position in the outer region, the length scale was taken to be the boundary layer thickness at 0.99 U_∞, although it was found that the profiles collapsed equally well when using the displacement thickness or momentum thickness. Regardless of the length scale used, the collapse is poor in the outer region for the profiles normalized by u_τ and U_∞. When the profiles are normalized by the proposed outer velocity scale, the collapse is much improved for $y/\delta > 0.07$ and for $650 < \delta^+ < 18 \times 10^3$.

CONCLUSIONS

A new scaling for the mean velocity profile of turbulent boundary layers was proposed. The new scaling leads to a power law for the overlap region of the mean velocity profile at low Reynolds numbers, and both a power law and log law region at high Reynolds numbers. Comparisons were made between the power law and 17 velocity profiles spanning a large range of Reynolds numbers ($650 < \delta^+ < 18 \times 10^3$ or $4.6 \times 10^3 < Re_\Theta < 58 \times 10^3$). This comparison showed that a power law with empirical constants determined from pipe flow data was in good agreement with boundary layer data, although large uncertainties in the friction velocity prevents us from making definitive conclusions. The proposed scaling requires a new outer velocity scale given by $U_\infty\delta^*/\delta$. The new outer velocity scale was used to normalize the 17 velocity profiles. This velocity scale collapsed the profiles significantly better than profiles normalized by the friction velocity or the freestream velocity. The comparison given in this paper supports the adoption of a new velocity scale for the outer region of turbulent boundary layers.

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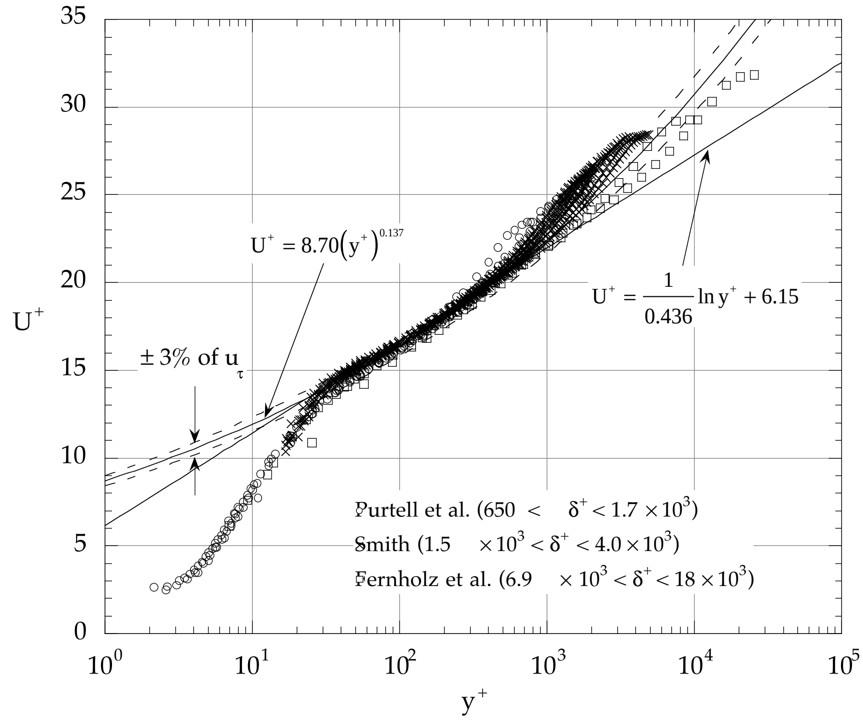

Figure 1. Velocity profiles normalized using inner scaling variables.

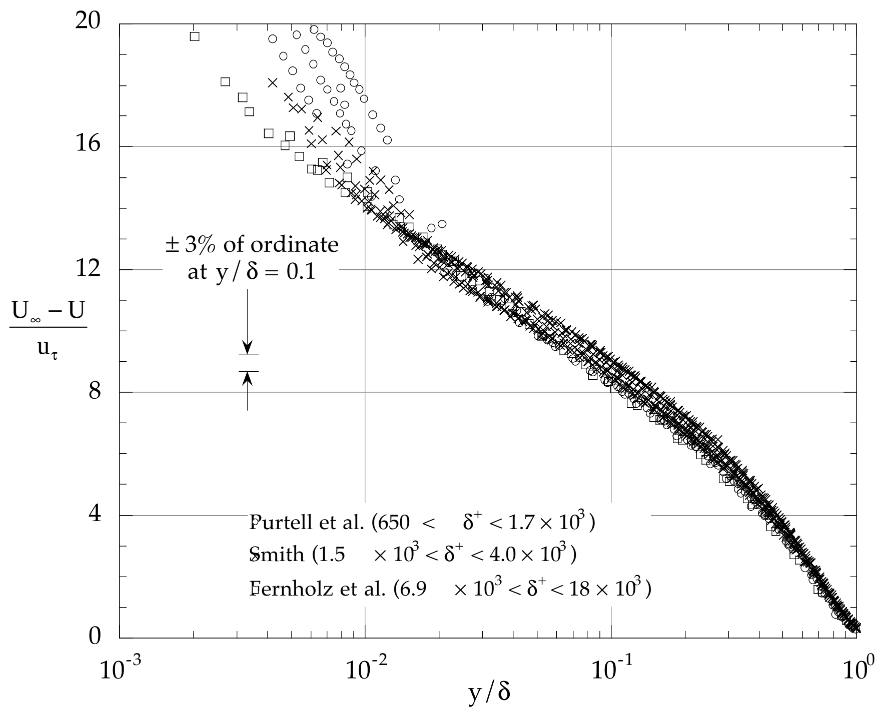

Figure 2. Velocity profiles normalized using the traditional outer scaling variables.

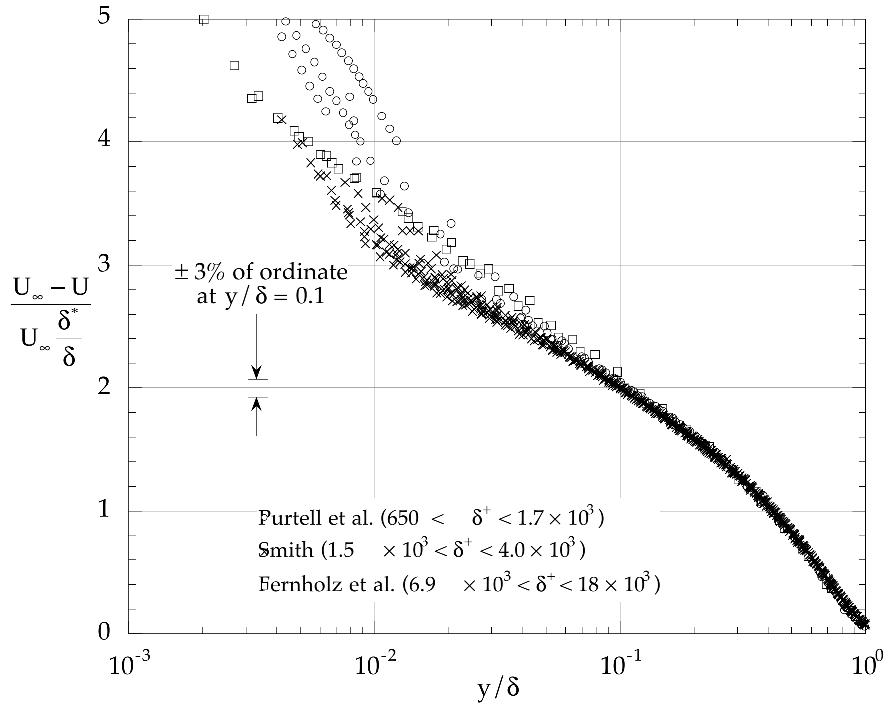

Figure 3. Velocity profiles normalized using the proposed outer scaling variables.

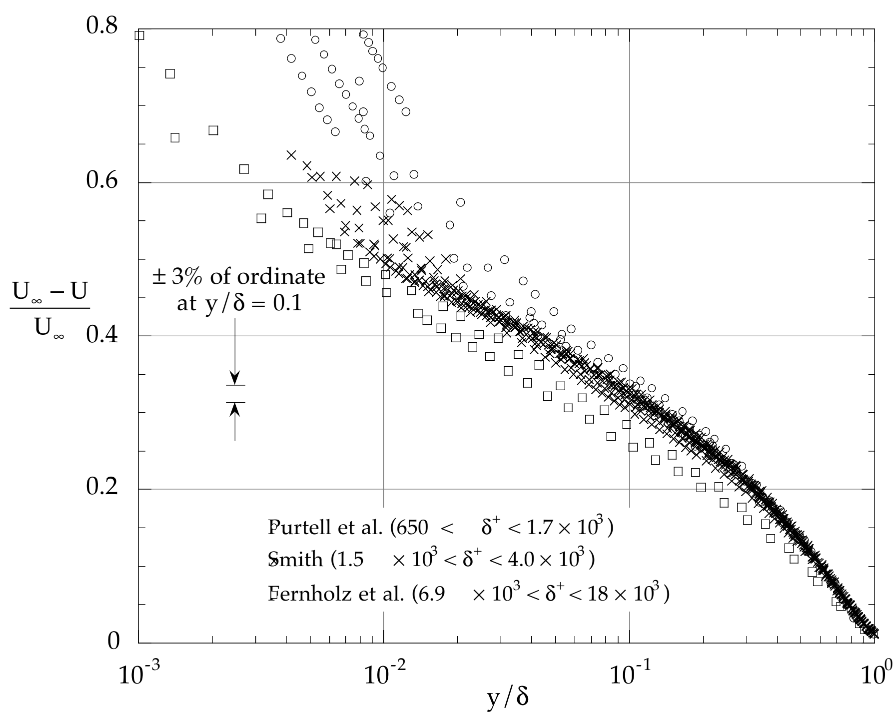

Figure 4. Velocity profiles normalized using the outer scaling variables proposed by George et al (1996).