The good and bad of liquidity risk

Thomas M. Eisenbach*
Princeton University

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Work in progress – comments welcome

Abstract

This paper models a firm optimally exposing itself to liquidity risk to overcome a risk shifting problem. The firm chooses a combination of short term and long term debt to finance a long term project. With only idiosyncratic uncertainty, the firm can use its maturity structure to implement an efficient policy. With added aggregate uncertainty this efficiency result breaks down as a wedge is driven between efficient policy and achievable policy. In a competitive equilibrium with multiple firms and endogenous liquidation values, a fire-sale externality leads to inefficiently high liquidity risk.

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1 Introduction

The development of the U.S. financial sector over the past decade has been characterized by several important trends. Financial innovations such as securitization, that promise a more efficient allocation of the economy’s capital resources have seen enormous growth. This growth has been accompanied by an increasing reliance of financial institutions on short-term market-based financing. At the same time, the consolidation of the financial sector since the repeal of the Glass-Steagall act in 1999 and a trend to individual diversification have increased the correlation of financial institutions portfolios and their exposure to aggregate uncertainty. In the recent crisis, these various elements interacted strongly and with serious consequences. After an aggregate shock originating in the housing market, liquidity in funding markets dropped significantly. This lead to serious problems for financial institutions having to roll over their short-term financing and saw the failure of big banks such as Bear Stearns and Lehman Brothers as well as many institutions in the shadow banking sector. There is a debate about whether the reliance on short-term debt was excessive and led to inefficient fire sales during the crisis.

In this paper, I present a model that seeks to address several of these phenomena. First, the model accounts for the increasing reliance on short-term debt as a reaction to the increasing liquidity of asset markets brought on by financial innovation. Next, the model shows how problematic this reliance on short-term debt is in the face of increasing exposure to aggregate uncertainty and that there is inefficient liquidation in case of a bad aggregate shock. Finally, the model shows that even without aggregate uncertainty, a fire-sale externality leads to inefficiently high liquidity risk in a competitive equilibrium.

The model is of a firm optimally choosing its debt maturity structure in the face of a risk shifting problem and two sources of uncertainty. The firm has an opportunity to invest in a project, facing idiosyncratic uncertainty about both the project’s final payoff as well as aggregate uncertainty about the liquidation value received when selling off the project’s assets to alternative uses. After the investment decision is made, additional information about the project’s expected payoff and the liquidation
value becomes available. Although the liquidation value can never recoup the initial investment, efficiency requires that the project be liquidated for sufficiently bad news about its expected payoff. Since the firm finances the investment by issuing debt, it is subject to a risk-shifting problem in the spirit of Jensen and Meckling (1976) and will never want to liquidate the project. Therefore the firm’s choice of maturity structure and the implied control rights play an important role for the resulting liquidation policy.

The firm can choose any combination of long-term and short-term debt to finance its investment. While long-term debt has the same maturity as the project’s final payoff, short-term debt has to be rolled over after additional information about the project’s expected payoff and the liquidation value becomes available. Liquidity risk can arise since it may not be possible to satisfy all withdrawals by short-term creditors, even by liquidating the entire project. Modelling the resulting coordination problem among short-term creditors as a global game, I derive a unique equilibrium with very intuitive properties. After bad news the short-term creditors withdraw their loans and the firm has to be liquidated while after good news all short-term creditors roll over and the project is continued.

The liquidity risk arising from this equilibrium is determined by the interaction of the two sources of uncertainty. A run on the firm can be triggered by bad news about the project’s expected payoff, or by bad news about alternative uses for the project’s assets, or both. In particular, given the maturity structure chosen at the time of investment, the firm will be less vulnerable to runs and therefore less likely to be liquidated if the liquidation value turns out to be high, than if it turns out to be low. The firm’s initial choice of maturity structure, i.e., how much of the project to finance with short-term as opposed to long-term debt, directly translates into how vulnerable the firm will be to liquidity risk when the additional information becomes available. The greater the fraction of short-term debt is, the greater will be the risk that the firm will suffer a run.

To distinguish between the different effects of the two sources of uncertainty on the firm’s maturity structure choice, I first analyze the case without aggregate uncertainty as a benchmark case. Without uncertainty about the alternative uses
for the project’s assets, the firm has full control over the amount of liquidity risk it exposes itself to. Since its creditors are competitive and have rational expectations, the firm receives the entire economic surplus of its investment opportunity, given the liquidation policy resulting from the maturity structure. Therefore the firm will choose its financing exactly so as to implement the efficient liquidation policy where the project is liquidated if and only if the expected payoff turns out to be less than the asset’s value in alternative uses. Optimally exposing itself to liquidity risk allows the firm to fully mitigate its moral hazard problem and achieve the first-best policy. This result has important implications for the comparative statics of the firm’s liquidity risk. If the project’s assets have a high value in alternative uses, efficiency calls for the project to be liquidated more often; the firm will therefore implement more liquidity risk when the anticipated liquidation value is higher. To do so, it has to chose a disproportionately greater fraction of short-term debt, to overcome the stabilizing effect the higher liquidation value has on the creditor interaction. Therefore, the model implies that increasing liquidation values over time will be accompanied by strong increases in the reliance on short-term financing.

Adding the aggregate uncertainty in form of a random liquidation value has two important effects. First, the efficient liquidation policy now depends on the realization of the liquidation value. If the liquidation value turns out to be high because there are valuable alternative uses for the project’s assets, efficiency calls for liquidation of projects that should be continued if the liquidation value were low. At the same time, the firm’s liquidity risk given the maturity structure chosen now varies with the realization of the liquidation value. If the liquidation value turns out to be high, creditors will be less worried about the firm’s liquidity, making a run less likely.

The key problem is that these two effects go in opposite directions. For high liquidation values, efficiency requires more projects to be liquidated but the firm’s increased stability leads to less liquidation. Vice versa, for low liquidation values, efficiency requires less projects to be liquidated but the firm’s decreased stability leads to more liquidation. Aggregate uncertainty effectively drives a wedge between the efficient liquidation policy and the achievable liquidation policy. The result is
inefficiently low liquidation in good aggregate states and inefficiently high liquidation in bad aggregate states. In choosing its maturity structure, the firm has to balance these two inefficiencies and will not be able to achieve the first-best policy. In particular, in the case of a bad aggregate shock with low liquidation values, this model implies widespread liquidation of projects with positive NPVs.

Finally, I extend the model without aggregate uncertainty to multiple firms facing a downward-sloping aggregate demand curve for their liquidated assets. This allows me to analyze the competitive equilibrium which jointly determines each firm’s maturity structure and the market value of liquidated assets. Since an individual firm doesn’t take into account its effect on the liquidation value of other firms’ assets, there is a fire-sale externality. The model shows that compared to the social optimum, in a competitive equilibrium firms choose a higher level of liquidity risk which leads to inefficiently high aggregate liquidation and depressed liquidation values.

**Related Literature** The role of short-term debt in the type of moral hazard setting of Jensen and Meckling (1976) has been discussed by Calomiris and Kahn (1991) and Rajan (1992). Short term creditor coordination problems are used in Diamond (2004) to commit creditors to costly enforcement. My paper differs from the existing literature mainly in the use of two sources of uncertainty. In particular, my model has a fully efficient outcome if only idiosyncratic uncertainty is present. The inefficiency arising with the addition of aggregate uncertainty arises from the wedge between efficient liquidation policy and achievable liquidation policy and not because inefficient liquidation is used as a threat to mitigate moral hazard. In a setting with contractual incompleteness, Diamond and Rajan (2000, 2001, 2005) show that short-term debt in the form of bank deposits is an important instrument in threatening liquidation and thereby overcoming renegotiation problems. Also related are several papers using a global game setting to model the coordination problem among creditors of a firm, starting with Morris and Shin (2004) who focus on the pricing of debt claims and the role of public information. Rochet and Vives (2004) use techniques similar to the ones in this paper to study lender-of-last-resort policies while Goldstein and Pauzner (2005) model the classic Diamond and Dybvig
(1983) bank run, solving the problem of multiple equilibria.

The rest of the paper is structured as follows. In Section 2 I present the setup of the model. Proceeding according to backward induction, in Section 3 I first analyze the coordination problem among short-term creditors. In Section 4 I then derive the optimal maturity structure for the case with idiosyncratic uncertainty and for the case with additional aggregate uncertainty. In Section 5 I extend the model to multiple firms and analyze the competitive equilibrium. Section 6 concludes.

2 Model

There are three time periods $t = 0, 1, 2$ and a single consumption good that can be stored without cost. All agents are risk neutral and have a discount rate of one.

**Project**  The firm has a project that requires an investment of 1 in the initial period $t = 0$ and has a random payoff in the final period $t = 2$ given by

$$
\begin{cases}
X \text{ with prob. } \theta, \\
0 \text{ with prob. } 1 - \theta.
\end{cases}
$$

In the interim period $t = 1$, the project can still be abandoned and any fraction of its assets can be sold off to alternative uses at a liquidation value of $\ell < 1$.

At the time of investment in $t = 0$, there is uncertainty about both the project’s expected payoff $\theta X$ and the liquidation value $\ell$, which is not resolved until additional information becomes available in the interim period $t = 1$. The key feature of this setup is that efficiency requires the project to be abandoned and its assets liquidated in $t = 1$, whenever the expected payoff turns out to be less than the liquidation value:

$$
\theta X < \ell
$$

The structure of the project and its timeline is illustrated in Figure 1.

I assume that the possible payoff $X$ is known in advance, but the success probability $\theta$ is distributed with c.d.f. $F$ on $[0, 1]$, while the liquidation value $\ell$ can take one of
two values $\ell_L, \ell_H$ with probabilities $p$ and $1 - p$ respectively, where $0 < \ell_L < \ell_H < 1$. The two random variables $\theta$ and $\ell$ are independent and the project has a positive ex-ante NPV:

$$\int_0^1 \theta X dF(\theta) - 1 > 0$$

The uncertainty about the project’s (expected) payoff captures the project’s idiosyncratic uncertainty inherent in, e.g., an R&D investment. The liquidation value, representing alternative uses for the project’s assets, is meant to capture aggregate uncertainty, e.g., about the state of the firm’s industry or the economy as a whole. Since the project’s ex-ante NPV is positive it is always efficient to invest initially. However, the efficiency of the continuation decision in the interim period depends both on the additional information about the expected final payoff as well as the alternative uses available for the project’s assets. I assume that the information that becomes available in $t = 1$ is not verifiable and cannot be contracted on in $t = 0$. Whether the continuation decision is made optimally will therefore depend on the firm’s financing and the resulting allocation of control rights.

**Financing** The firm has no funds of its own and has to raise the entire investment amount of 1 through loans from competitive outside investors in $t = 0$. Since we always have $\ell < 1$, the proceeds in case of liquidation go entirely to the firm’s creditors. Therefore the firm’s owners receive a payoff only if the project is not liquidated in $t = 1$ and is successful in $t = 2$. This payoff structure implies the
standard risk-shifting problem of equity holders as in Jensen and Meckling (1976) since they always prefer continuing the project and never want to liquidate early.

I assume that the firm can choose any combination of long-term and short-term debt to finance its project. Long term debt matures in the final period $t = 2$ at a face value of $R_L$. Short term debt, on the other hand, has to be rolled over in the interim period $t = 1$ and, if rolled over, has a face value of $R_S$ in $t = 2$. Instead of rolling over in $t = 1$ a short-term creditor has the right to withdraw the principal of his loan.\(^1\) This creates the possibility of the firm becoming illiquid in $t = 1$ since it may face more withdrawals from short-term creditors than it can satisfy by liquidating even the entire project.

Denoting by $\alpha \in [0, 1]$ the fraction of the project financed by short-term debt, the firm’s choice of debt maturity structure in the initial period $t = 0$ amounts to a combination of short-term and long-term debt $(\alpha, 1 - \alpha)$. The interest rates $R_L$ and $R_S$ are then determined endogenously, taking into account both the exogenous idiosyncratic and aggregate risk, as well as the the liquidity risk arising endogenously from the maturity structure.

\textbf{Liquidity Risk} Denoting the fraction of short-term creditors who withdraw their loans in $t = 1$ by $\lambda$, the firm has to liquidate enough of the project to raise $\alpha \lambda$ for repayment. Since the firm can raise at most $\ell$ by liquidating the entire project, it \textit{can} become illiquid in $t = 1$ if $\alpha > \ell$ and it \textit{will} be illiquid whenever\(^2\)

$$\lambda > \frac{\ell}{\alpha}.$$  

If the firm becomes illiquid in $t = 1$, there will be nothing left in $t = 2$ to repay any short-term creditors who decided to roll over their loan. The short-term creditors therefore face a coordination problem which I model as a global game by assuming

\(^1\)The assumption that the interim face value equals the principal is just a normalization and without loss of generality.

\(^2\)The paper focuses on the case of $\alpha > \ell$ where the firm can become illiquid. Appendix C discusses the case of $\alpha \leq \ell$ and presents sufficient conditions for the optimal maturity structure to satisfy $\alpha > \ell$. 

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a small amount of noise in each creditor’s information. The firm’s liquidity risk is derived from the equilibrium of the creditors’ coordination game.

3 Interim creditor coordination

The short-term debt in this model is market based financing such as commercial paper. In these markets, most of the funds are allocated through intermediaries, e.g., money market funds in the commercial paper market. I therefore assume that the roll-over decision is taken by a fund manager on behalf of the actual investor.\footnote{This assumption of intermediation in the supply of short term funding is similar to Rochet and Vives (2004). In our model it serves two purposes: First, it allows to study spillover effects of the incentive structures in the financial industry onto the firms receiving funding. Secondly, it simplifies the analysis and exposition without affecting the validity of the results. Appendix B presents the model without intermediation.}

There is a continuum of fund managers with the following payoffs. If a fund manager withdraws his loan in $t = 1$ he receives a constant payoff of $Y > 0$, a base salary. If the fund manager rolls over his loan in $t = 1$ the payoff depends on whether the firm repays the loan in $t = 2$: if the firm repays, the fund manager receives a payoff of $bY$, his base salary multiplied by a bonus factor $b > 1$; if the firm doesn’t repay, the fund manager receives a payoff of zero. These payoffs capture the crucial difference between the two actions the fund manager has available. Withdrawing the loan is the “safe” action: even if the firm fails and the loan is not repaid in full, the fund manager’s clients (the actual investors) can’t blame him. Rolling over the loan is the “risky” action: the fund manager will be blamed if the clients lose their money but will receive his bonus if all goes well. A higher $b$ therefore corresponds to higher-powered incentive structures and implies that the fund managers are willing to take greater risks. It can also be interpreted as a proxy for the risk tolerance of the short-term funding sector in general.\footnote{See Krishnamurthy (2010) for a discussion of financial institutions’ risk tolerance.}

Each fund manager has to make his roll-over decision in the interim period $t = 1$ based on the following information. While the resolution of aggregate uncertainty in the form of the liquidation value $\ell$ is perfectly observed by everyone and becomes
common knowledge, the resolution of idiosyncratic uncertainty in the form of the success probability \( \theta \) is not perfectly observed. Each fund manager \( i \) receives a noisy signal \( \sigma_i = \theta + \varepsilon_i \), where the signal noise \( \varepsilon_i \) is i.i.d. uniformly on \( [-\varepsilon, \varepsilon] \) for some arbitrarily small \( \varepsilon > 0 \).

If the firm is vulnerable to become illiquid because of a liquidation value \( \ell < \alpha \), each fund manager’s expected payoff of rolling over depends critically on the fraction \( \lambda \) of other fund managers who withdraw. Since he only receives the bonus \( bY \) if the firm remains liquid in \( t = 1 \) and the project succeeds in \( t = 2 \), the expected payoff of rolling over is

\[
\Pr[\text{liquid}|\ell, \sigma_i] \cdot \Pr[\text{success}|\ell, \sigma_i] \cdot bY
\]

\[
= \Pr\left[ \lambda \leq \frac{\ell}{\alpha} \big| \ell, \sigma_i \right] \cdot E[\theta|\ell, \sigma_i] \cdot bY,
\]

while the payoff to withdrawing is \( Y \) for sure.

Using standard global games techniques we can derive the unique equilibrium of the fund managers’ coordination game, which is in symmetric switching strategies such that each fund manager withdraws for all signals below a threshold and rolls over for all signals above.\(^5\) The equilibrium switching point is determined by the fact that a fund manager exactly at the switching point has to be indifferent between rolling over and withdrawing, given his belief about the fraction \( \lambda \) of others withdrawing. Taking the limit as the signal noise \( \varepsilon \) goes to zero, the distribution of \( \lambda \) conditional on being at the switching point \( \hat{\theta} \) becomes uniform on \([0, 1]\). The indifference condition for a fund manager at the switching point therefore simplifies to

\[
\frac{\ell}{\alpha} \cdot \hat{\theta} \cdot bY = Y,
\]

\(^5\)Appendix A discusses the global game of this paper in more detail. For a comprehensive discussion of the use of global games since the seminal papers of Carlsson and van Damme (1993a,b) see Morris and Shin (2003).
which pins down the switching point as

\[ \hat{\theta} = \frac{\alpha}{\ell b} \]

(1)

**Proposition 1** Given a realization of \( \ell < \alpha \) and for \( \varepsilon \to 0 \), the unique equilibrium among short-term creditors is in switching strategies around the success probability threshold \( \hat{\theta} = \frac{\alpha}{\ell b} \):

- For realizations of \( \theta \) below \( \hat{\theta} \), all short-term creditors withdraw their loans, resulting in a run that causes the firm to become illiquid.
- For realizations of \( \theta \) above \( \hat{\theta} \) all short-term creditors roll over their loans and the firm remains liquid.

The simple structure of the equilibrium threshold highlights the key determinants of the firm’s liquidity risk in \( t = 1 \), given by \( \Pr[\theta \ell < \alpha/b] \). First, liquidity risk is increasing in the fraction of short-term debt \( \alpha \). Having a balance sheet that relies more heavily on short-term debt makes the firm more vulnerable to runs since it increases the total amount of withdrawals the firm may face. Second, liquidity risk is decreasing in the short-term fund managers’ bonus \( b \). Fund managers more willing to take a risk by rolling over their loan make the firm less vulnerable to runs. This implies that, e.g., regulation which reduces the risk tolerance of institutions providing short-term funding, increases the liquidity risk of borrowing firms.\(^6\)

Besides these two deterministic factors, whether the firm suffers a run or not depends on both sources of uncertainty, idiosyncratic and aggregate. A run can be triggered by bad news about the project’s expected payoff, or by bad news about the alternative uses for the project’s assets, or both. Note that the two effects interact, so the firm’s idiosyncratic liquidity risk is decreasing in the liquidation value. A lower liquidation value makes the firm more vulnerable to runs since there is less mass to satisfy withdrawals from.

\(^6\)For a discussion of the differences between micro- and macro-prudential regulation, see Morris and Shin (2008).
It is important to keep in mind that the fraction $\alpha$ of short-term debt is set endogenously by the firm in $t = 0$, in anticipation of the resulting liquidity risk. The next section derives the firm’s optimal decision and discusses the different effects of idiosyncratic and aggregate risk.

4 Ex-ante maturity structure

In the initial period $t = 0$, short-term and long-term investors as well as the firm anticipate what will happen in the following periods. This means that the interest rates on short-term and long-term debt, $R_S$ and $R_L$, have to compensate investors for the riskless rate of return $R$. The firm, when choosing its debt maturity structure $(\alpha, 1 - \alpha)$, takes into account the effect of $\alpha$ on the interest rates $R_S$ and $R_L$, as well as the effect on the creditor coordination in $t = 1$.

To highlight the different effects of idiosyncratic risk and aggregate risk in the present model, in Section 4.1 I first derive the firm’s optimal maturity structure if there is no aggregate uncertainty. Then, in Section 4.2, I analyze the added complications aggregate uncertainty introduces and derive the firm’s optimal maturity structure with aggregate uncertainty.

4.1 Only idiosyncratic uncertainty

Without aggregate uncertainty, potential alternative uses for the firm’s assets in $t = 1$ are deterministic and given by the fixed liquidation value $\ell$. The only uncertainty stems from the project’s payoff and this uncertainty is partially resolved in $t = 1$ when the success probability $\theta$ is drawn from its distribution $F$. Depending on the additional information received about the project’s expected payoff, it will be efficient to either continue with the project or to abandon it and put the liquidated assets to alternative use. Liquidation is efficient whenever the project’s expected payoff is less than the liquidation value

$$\theta X < \ell.$$

To set up the firm’s maximization problem it is instructive to first derive the
endogenous interest rates $R_S$ and $R_L$. Since the liquidation value is deterministic, so is the threshold determining the outcome of the creditor coordination in $t = 1$:

$$\hat{\theta} = \frac{\alpha}{\ell b}$$

For realizations of $\theta$ below $\hat{\theta}$, there will be a creditor run on the firm. In this case, each short-term creditor receives an equal share of the liquidation proceeds, $\ell/\alpha$, while long-term creditors don’t receive anything. For realizations of $\theta$ above $\hat{\theta}$, all short-term creditors roll over their loans and the firm continues to operate the project. In this case, all creditors receive the face value of their loan in $t = 2$, if the project is successful. Note that we are now dealing with the payoffs of the actual investors whose money is at stake, not the payoffs of the fund managers.\(^7\)

For a short-term creditor this implies an ex-ante expected payoff given by

$$F(\hat{\theta})\frac{\ell}{\alpha} + \int_{\hat{\theta}}^{1} \theta R_S dF(\theta).$$

With probability $F(\hat{\theta})$ there is a run on the firm leading to full liquidation; in this case the short-term creditor receives an equal share $\ell/\alpha$ of the liquidation value. If there is no run, the short-term creditor receives the face value of his loan $R_S$ if the project is successful which happens with probability $\theta$; this is the case for all realizations $\theta \in (\hat{\theta}, 1]$. A long-term creditor, on the other hand, only receives a payment if there is no run in the interim period and the project is successful in the final period. The ex-ante expected payoff for long-term debt therefore is

$$\int_{\hat{\theta}}^{1} \theta R_L dF(\theta).$$

Since all creditors have to be compensated for the riskless rate of return $R^*$, the

\(^7\)It is natural to assume that the payments to the fund manager, the bonus $bY$ and base salary $Y$, have to be paid by the investor. For simplicity we assume that these payments are negligible as a fraction of total investment and focus on the limiting case of $Y \rightarrow 0$ but holding $b$ constant.
endogenous interest rates for short-term and long-term debt are given by

\[ R_S = \frac{R^* - F(\hat{\theta})}{\int_{\hat{\theta}}^1 \theta dF(\theta)} \quad \text{and} \quad R_L = \frac{R^*}{\int_{\hat{\theta}}^1 \theta dF(\theta)}. \]  

(2)

Given the liquidity risk and interest rates for a given maturity structure \((\alpha, 1 - \alpha)\), it remains to derive the firm’s ex-ante payoff. If there is no run by short-term creditors in the interim period \((\theta > \hat{\theta})\), and the project is successful in the final period, the firm’s payoff is

\[ X - \alpha R_S - (1 - \alpha) R_L. \]

It receives the successful project’s cash flow and has to repay the face value of its short-term and long-term debt. If there is a run in \(t = 1\) or the project is not successful in \(t = 2\) the firm’s payoff is zero. The ex-ante expected payoff of the firm therefore is

\[ \int_{\hat{\theta}}^1 \theta [X - \alpha R_S - (1 - \alpha) R_L] dF(\theta). \]

Substituting in the interest rates from (2) and rearranging, the firm’s ex-ante expected payoff becomes

\[ F(\hat{\theta})\ell + \int_{\hat{\theta}}^1 \theta X dF(\theta) - R^*. \]

Due to the rational expectations and competitive creditors, the firm receives the entire economic surplus of its project, given the liquidity-risk threshold \(\hat{\theta}\). Recalling the expression for \(\hat{\theta}\) from the creditor coordination game in (1), the firm chooses \(\alpha\) to maximize

\[ F(\hat{\theta})\ell + \int_{\hat{\theta}}^1 \theta X dF(\theta) - R^* \quad \text{subject to} \quad \hat{\theta} = \frac{\alpha}{\ell b}. \]

In choosing its maturity structure \(\alpha\), the firm effectively chooses a liquidity-risk threshold \(\hat{\theta}\). The first order condition to the firm’s problem is

\[ f(\hat{\theta}) \frac{1}{\ell b} (\ell - \hat{\theta} X) = 0, \]

which implies the following result.
Proposition 2 Without aggregate uncertainty, the firm chooses a maturity structure to implement the efficient liquidation policy:

\[ \alpha^* = \frac{\ell^2 b}{X} \quad \text{and} \quad \hat{\theta}^* = \frac{\ell}{X}. \]

With an equity payoff ex post the firm is subject to a classic risk shifting problem and would never want to liquidate in the interim period. The firm therefore uses the maturity structure as a commitment device to implement a liquidation threshold \( \hat{\theta} \) maximizing its payoff. Optimally exposing itself to liquidity risk then allows the firm to fully mitigate the moral hazard problem and maximize its project’s value.

This result has important implications for the comparative statics of the firm’s liquidity risk. While the liquidity-risk threshold \( \hat{\theta} \) for a given maturity structure \( \alpha \) is decreasing in the liquidation value \( \ell \), the optimal liquidity-risk threshold \( \hat{\theta}^* \) is increasing in the liquidation value \( \ell \). As discussed in Section 3 above, for a given maturity structure, a higher liquidation value has a stabilizing effect on the creditor coordination, making a run less likely and therefore reducing liquidity risk. From an ex-ante point of view, however, efficiency requires a higher expected payoff for the project when there are better alternative uses for the project’s assets. Since the firm is able to implement the efficient liquidation policy, a higher liquidation value will cause it to increase liquidity risk.

The fraction of short-term debt \( \alpha^* \) in the optimal maturity structure reflects both these effects of the liquidation value. Since a higher liquidation value makes the firm more stable while efficiency calls for more liquidation, the fraction of short-term debt is convex in \( \ell \). In order to achieve efficiency with a higher liquidation value, the firm has to overcompensate for the stabilizing effect by choosing an disproportionately higher fraction of short-term debt.

This result of the firm being able to achieve full efficiency by optimally exposing itself to liquidity risk serves as a benchmark for the case with aggregate uncertainty discussed next.
4.2 Idiosyncratic and aggregate uncertainty

With aggregate uncertainty, potential alternative uses for the firm’s assets may vary depending on exogenous factors, resulting in a random liquidation value

$$\ell = \begin{cases} \ell_H & \text{with prob. } p, \\ \ell_L & \text{with prob. } 1 - p. \end{cases}$$

This additional source of uncertainty has two implications. The first implication is that the efficient project continuation decision may be different depending on the realization of $\ell$. While in the case without aggregate uncertainty there was a single critical value for the project’s expected payoff, there are now two. For the low liquidation value $\ell_L$ the project should only be continued if $\theta X > \ell_L$, while for the high liquidation value $\ell_H$ the condition is $\theta X > \ell_H$. In particular, for realizations of the project’s success probability $\theta$ in the interval $[\ell_L/X, \ell_H/X]$, efficiency calls for liquidation if the assets have high-valued alternative uses ($\ell = \ell_H$) and for continuation if the assets have low-valued alternative uses ($\ell = \ell_L$).

The second implication of aggregate uncertainty is that the creditor coordination game will be different depending on the realization of $\ell$. There are now two equilibrium switching points, $\hat{\theta}_H$ and $\hat{\theta}_L$, one for each realization of $\ell$:

$$\hat{\theta}_H = \frac{\alpha}{\ell_Hb} \quad \text{and} \quad \hat{\theta}_L = \frac{\alpha}{\ell_Lb}$$

If the liquidation value turns out to be high ($\ell = \ell_H$), each creditor is less concerned about the other creditors withdrawing their loans and therefore more willing to roll over his loan than when the liquidation value turns out to be low ($\ell = \ell_L$). Therefore the firm will be more stable and less likely to suffer a run by its short-term creditors if the liquidation value is high, which is reflected in the liquidity risk threshold being smaller than when the liquidation value is low,

$$\hat{\theta}_H < \hat{\theta}_L.$$ 

This implies that the project will be liquidated less often when the liquidation value
is high, which runs against the policy required by efficient liquidation. At the same time, the project will be liquidated more often when the liquidation value is low, which again runs against the policy required by efficient liquidation. These two effects are illustrated in Figure 2.

As in the case without aggregate uncertainty, the firm receives the entire economic surplus of its project, given the liquidation resulting from its maturity structure. The firm therefore chooses $\alpha$ to maximize

$$p \left[ F(\hat{H})\ell_H + \int_{\hat{H}}^{1} \theta XdF(\theta) \right] + (1-p) \left[ F(\hat{L})\ell_L + \int_{\hat{L}}^{1} \theta XdF(\theta) \right] - R^*$$

subject to $\hat{H} = \frac{\alpha}{\ell_H b}$ and $\hat{L} = \frac{\alpha}{\ell_L b}$.

The first order condition to the firm’s problem is now

$$p \left[ f \left( \frac{\alpha}{\ell_H b} \right) \frac{1}{\ell_H b} \left( \ell_H - \frac{\alpha}{\ell_H b} X \right) \right] + (1-p) \left[ f \left( \frac{\alpha}{\ell_L b} \right) \frac{1}{\ell_L b} \left( \ell_L - \frac{\alpha}{\ell_L b} X \right) \right] = 0,$$

which cannot be solved explicitly without specifying a functional form for the density of idiosyncratic uncertainty $f$. For simplicity, I will assume the distribution to be uniform and get the following result.
Proposition 3 With aggregate uncertainty and \( F \) uniform, the firm chooses a maturity structure

\[
\alpha^* = \frac{\ell_H^2 \ell_L^2}{p\ell_L^2 + (1 - p) \ell_H^2} \frac{b}{X} = \xi \frac{\ell_H^2 b}{X} + (1 - \xi) \frac{\ell_L^2 b}{X}
\]

with \( \xi := \frac{p\ell_L^2}{p\ell_L^2 + (1 - p) \ell_H^2} \),

which results in the liquidation policies

\[
\dot{\theta}_H = \frac{\ell_L^2}{p\ell_L^2 + (1 - p) \ell_H^2} \frac{\ell_H}{X} \quad \text{and} \quad \dot{\theta}_L = \frac{\ell_H^2}{p\ell_L^2 + (1 - p) \ell_H^2} \frac{\ell_L}{X}.
\]

The overall effect of aggregate uncertainty is that it drives a wedge between the efficient liquidation policy and the achievable liquidation policy. The effectiveness of using the maturity structure to mitigate moral hazard and implement an efficient liquidation policy is hampered when aggregate uncertainty is added to the firm’s idiosyncratic uncertainty. It is important to note that there are efficiency losses for both realizations of the liquidation value since \( \hat{\theta}_H < \ell_H/X \) and \( \hat{\theta}_L > \ell_L/X \). When the liquidation value is high, projects that should be liquidated are continued while when the liquidation value is low, projects that should be continued are liquidated. The firm tries to balance the efficiency losses in the two cases and chooses fraction of short-term debt for its maturity structure that is a convex combination of the optimal fractions if the liquidation value were known to be \( \ell_H \) and \( \ell_L \) respectively.

5 Multi-firm equilibrium and fire sales

Having established and analyzed the model for a single firm I now extend it to the case of multiple firms, thereby endogenizing the liquidation value \( \ell \). This allows me to compare the allocation arising in a competitive equilibrium with the allocation a social planner would choose and highlight the inefficiency caused by a fire-sale externality.

There is a continuum of identical firms \( j \in [0, 1] \), each with an investment op-
portunity in $t = 0$ as before and the success probabilities $\{\theta_j\}$ are drawn i.i.d. from the distribution $F$ in $t = 1$. The key assumption is a downward-sloping aggregate demand for liquidated assets in the interim period $t = 1$. Each individual firm $j$ still faces a perfectly elastic demand for its assets but the liquidation value depends on the total mass of assets $\phi \in [0,1]$ sold off by all firms and is given by the inverse demand function $\ell(\phi)$ with

$$\ell(\phi) \in (0, 1), \quad \ell'(\phi) < 0 \quad \text{for all} \quad \phi \in [0,1].$$

Such a decreasing liquidation value could originate from decreasing marginal productivity of the assets in alternative uses or from cash-in-the-market pricing. For this section I focus on the case without aggregate uncertainty. As will become clear below, this is guaranteed by the assumption of i.i.d. $\theta_j$s and a deterministic $\ell(\phi)$.

### 5.1 Competitive equilibrium

I first derive the individual firm $j$'s optimal maturity structure, taking aggregate asset liquidation $\phi$ as given, and then derive the equilibrium which jointly determines the values of $\phi$ and $\{\alpha_j\}$.

The short term creditors of an individual firm $j$ take the liquidation value $\ell(\phi)$ as given in their coordination game in $t = 1$, which results in a switching point as in Section 3:

$$\hat{\theta}_j = \frac{\alpha_j}{\ell(\phi)b}.$$

Firm $j$ also takes $\ell(\phi)$ as given when deciding on its maturity structure in $t = 0$, choosing $\alpha_j$ to maximize

$$F(\hat{\theta}_j)\ell(\phi) + \int_{\hat{\theta}_j}^{1} \theta X dF(\theta) - R^* \quad \text{subject to} \quad \hat{\theta}_j = \frac{\alpha_j}{\ell(\phi)b}.$$

The optimal fraction of short term debt and the implemented liquidation policy
respectively, are analogous to Section 4.1:

\[ \alpha_j^* = \frac{\ell(\phi)^2 b}{X} \quad \text{and} \quad \hat{\theta}_j^* = \frac{\ell(\phi)}{X} \]

Since all firms are symmetric ex ante, the equilibrium is symmetric with \( \alpha_j^* = \alpha_k^* \)
and \( \hat{\theta}_j^* = \hat{\theta}_k^* \) for all \( j, k \). With a continuum of firms and the i.i.d. success probabilities \( \{\theta_j\} \), the total mass of assets sold off in \( t = 1 \) is equal to the fraction of firms with \( \theta_j \leq \hat{\theta}^* \) who experience a run by their short term creditors and are liquidated. The competitive equilibrium value \( \phi^{CE} \) is therefore the solution to the fixed-point equation

\[ \phi^{CE} = F\left(\hat{\theta}^* \left( \phi^{CE} \right) \right). \]

Since \( \frac{d\hat{\theta}^*}{dt} > 0 \), there is a unique solution for \( \phi^{CE} \).

**Proposition 4** The competitive equilibrium is characterized by a fraction \( \phi^{CE} \) of firms liquidated, implicitly defined by

\[ \phi^{CE} = F\left(\frac{\ell(\phi^{CE})}{X} \right), \]

as well as optimal maturity structures \( \{\alpha_j^{CE}\} \) and resulting liquidation thresholds \( \{\hat{\theta}_j^{CE}\} \) given by

\[ \alpha_j^{CE} = \frac{\ell(\phi^{CE})^2 b}{X} \quad \text{and} \quad \hat{\theta}_j^{CE} = \frac{\ell(\phi^{CE})}{X} \quad \text{for all} \quad j \in [0,1]. \]

In the competitive equilibrium, the individual firm doesn’t take into account the effect its own maturity structure has on the liquidation value of other firms’ assets. This is an example of a fire-sale externality where one firm’s asset liquidation depresses the liquidation values for other firms. The next section shows the effect of this externality in the present model by comparing the competitive equilibrium to a social-planner optimum.
5.2 Fire-sale externality

In contrast to the individual firms, a social planner takes into account the effect the liquidation policy has on the liquidation value. In particular, the social planner anticipated that for an implemented liquidation threshold \( \hat{\theta} \), the mass of assets liquidated and put to alternative uses is \( \phi = F(\hat{\theta}) \). The social planner therefore chooses \( \hat{\theta} \) to maximize the representative firm’s expected economic surplus,

\[
F(\hat{\theta})\ell(\phi) + \int_{\hat{\theta}}^{1} \theta X dF(\theta) - R^s, \quad \text{subject to} \quad \phi = F(\hat{\theta}).
\]

The socially optimal liquidation threshold \( \hat{\theta}^{SO} \) is therefore defined by the first-order condition

\[
\ell(F(\hat{\theta}^{SO})) - \hat{\theta}^{SO} X + F(\hat{\theta}^{SO}) \ell'(F(\hat{\theta}^{SO})) = 0. \tag{3}
\]

Comparing equation (3) to the corresponding equation defining the competitive equilibrium threshold \( \hat{\theta}^{CE} \),

\[
\ell(F(\hat{\theta}^{CE})) - \hat{\theta}^{CE} X = 0,
\]

the two differ in the part \( F(\hat{\theta})\ell'(F(\hat{\theta})) < 0 \) which captures the effect of the liquidation policy on the liquidation value. Since \( \frac{d}{d\theta}[\ell(F(\hat{\theta})) - \hat{\theta} X] < 0 \) this implies the following result.

**Proposition 5** Compared to the competitive equilibrium (CE), the social optimum (SO) has a lower liquidity risk threshold, \( \hat{\theta}^{SO} < \hat{\theta}^{CE} \), which implies less aggregate liquidation, \( \phi^{SO} < \phi^{CE} \), and therefore a higher liquidation value \( \ell^{SO} > \ell^{CE} \).

The extension of the model to multiple firms highlights the fire-sale externality as another source of inefficiency in the use of short-term debt. Given the assumptions in this section of i.i.d. success probabilities and a deterministic demand for liquidated assets, there is no aggregate uncertainty. This means that each individual firm can use liquidity risk to optimally overcome its risk-shifting problem, given the aggregate liquidation value.

However, once the aggregate liquidation value is endogenized through a decreasing demand for liquidated assets, the individually optimal liquidity risk leads to
inefficiently high aggregate liquidation and depressed liquidation values. This inefficiency is caused by the negative pecuniary externality that one firm’s liquidation has on the liquidation value of the other firms’ assets. Since an individual firm doesn’t take this fire-sale externality into account, it chooses more liquidity risk than a social planner would.

6 Conclusion

In this paper I study the different effects of short-term debt and the liquidity risk it causes. The benchmark model of a single firm facing only idiosyncratic risk establishes the mechanism of using the deb-maturity structure to overcome a risk-shifting problem and implement an optimal liquidation policy. By anticipating the coordination problem among short-term creditors, the firm can choose the right amount of liquidity risk to maximize its economic surplus.

The introduction of aggregate uncertainty, however, severely affects this mechanism and drives a wedge between desired and achievable liquidation policy. This implies that in bad aggregate states there is too much liquidation, while in good aggregate states there is too little liquidation.

The extension of the model to multiple firms and an endogenous liquidation value highlights a second source of inefficiency, even without aggregate uncertainty. Since the individual firm doesn’t take into account the fire-sale externality it imposes on other firms, the competitive equilibrium will have inefficiently high levels of liquidation and depressed liquidation values.
Appendix

A Global game

To apply the standard global games results summarized by Morris and Shin (2003) we first need an upper and lower dominance region. If the realization $\theta$ is sufficiently low such that $\theta bY < Y$, then withdrawing the loan is a dominant action regardless of how many others withdraw. Let us denote the boundary of the lower dominance region by $\underline{\theta} := 1/b$. On the other hand, if the realization $\theta$ is sufficiently high we take the approach of Goldstein and Pauzner (2005) and assume that the firm cannot become illiquid, e.g. because the project matures early and pays off $X$ for sure, thus making rolling over the loan a dominant action. Let us denote the boundary for the upper dominance region by $\overline{\theta} < 1$. For all realizations $\theta \in [\underline{\theta}, \overline{\theta}]$ there is a coordination problem among fund managers since the relative payoff of withdrawing and rolling over depends on the fraction $\lambda$ of others withdrawing.

Given the signal structure, for a realization $\theta$ the distribution of signals is uniform on $[\theta - \varepsilon, \theta + \varepsilon]$ and for a signal $\sigma$ the conditional distribution of $\theta$ is

$$f(\theta | \sigma) = \begin{cases} \frac{f(\theta)}{F(\sigma+\varepsilon)-F(\sigma-\varepsilon)} & \text{for } \theta \in [\sigma - \varepsilon, \sigma + \varepsilon], \\ 0 & \text{otherwise.} \end{cases}$$

In equilibrium, a fund manager with signal $\sigma_i = \hat{\sigma}$ has to be indifferent between rolling over and withdrawing:

$$\Pr \left[ \lambda \leq \frac{\ell}{\alpha} \bigg| \hat{\sigma} \right] E [\theta | \hat{\sigma}] bY = Y$$

Given the distribution of $\theta | \hat{\sigma}$ we have an expression for $E [\theta | \hat{\sigma}]$ so it remains to derive the distribution of $\lambda | \hat{\sigma}$. We can derive the c.d.f. $G(\lambda | \hat{\sigma})$ as follows: the probability that a fraction less than $\lambda$ receives a signal less than $\hat{\sigma}$ (and therefore withdraws)
equals the probability that $\theta$ is greater than $\theta'$ defined by

$$
\frac{\hat{\sigma} - (\theta' - \varepsilon)}{2\varepsilon} = \lambda
$$

$$
\Rightarrow \theta' = \hat{\sigma} + \varepsilon - 2\varepsilon \lambda
$$

We therefore have

$$
G (\lambda|\hat{\sigma}) = 1 - F (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda|\hat{\sigma})
$$

$$
= 1 - \int_{\hat{\sigma} - \varepsilon}^{\hat{\sigma} + \varepsilon - 2\varepsilon \lambda} \frac{f (\theta)}{F (\hat{\sigma} + \varepsilon) - F (\hat{\sigma} - \varepsilon)} d\theta
$$

$$
= \frac{F (\hat{\sigma} + \varepsilon) - F (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda)}{F (\hat{\sigma} + \varepsilon) - F (\hat{\sigma} - \varepsilon)}.
$$

It remains to derive the limits as the signal noise $\varepsilon$ goes to zero. First, we have that

$$
\lim_{\varepsilon \to 0} E [\theta|\hat{\sigma}] = \hat{\sigma}.
$$

Secondly, we have that

$$
\lim_{\varepsilon \to 0} G (\lambda|\hat{\sigma}) = \lim_{\varepsilon \to 0} \frac{F (\hat{\sigma} + \varepsilon) - F (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda)}{F (\hat{\sigma} + \varepsilon) - F (\hat{\sigma} - \varepsilon)}
$$

$$
= \lim_{\varepsilon \to 0} \frac{f (\hat{\sigma} + \varepsilon) - f (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda) (1 - 2\lambda)}{f (\hat{\sigma} + \varepsilon) + f (\hat{\sigma} - \varepsilon)}
$$

$$
= \lim_{\varepsilon \to 0} \frac{f (\hat{\sigma} + \varepsilon) - f (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda)}{f (\hat{\sigma} + \varepsilon) + f (\hat{\sigma} - \varepsilon)} + \lim_{\varepsilon \to 0} \frac{2\lambda f (\hat{\sigma} + \varepsilon - 2\varepsilon \lambda)}{f (\hat{\sigma} + \varepsilon) + f (\hat{\sigma} - \varepsilon)}
$$

$$
= \frac{0}{2f (\hat{\sigma})} + \frac{2\lambda f (\hat{\sigma})}{2f (\hat{\sigma})}
$$

$$
= \lambda
$$

So the distribution of $\lambda$ conditional on being at the switching point becomes uniform as the signal noise goes to zero.
B Model without intermediation

Instead of assuming fund manager payoffs, we can work with the real creditor payoffs. They are given by

<table>
<thead>
<tr>
<th>roll over</th>
<th>withdraw</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha \lambda \leq \ell )</td>
<td>( \theta R_S )</td>
</tr>
<tr>
<td>( \alpha \lambda &gt; \ell )</td>
<td>1</td>
</tr>
</tbody>
</table>

If the firm doesn’t fail, a creditor who withdraws receives 1, the principal of his loan and a creditor who rolls over receives the face value \( R_S \) of his loan with probability \( \theta \). If the firm fails, a creditor who withdraws receives an equal share of the liquidation value, \( \frac{\ell}{\alpha \lambda} \), and a creditor who rolls over receives nothing.

With signal noise going to zero, indifference at the switching point requires

\[
\frac{\ell}{\alpha} \theta R_S = \frac{\ell}{\alpha} + \int_0^1 \frac{\ell}{\alpha \lambda} d\lambda
\]

so the critical value is given by

\[
\hat{\theta} = \frac{1}{R_S} \left(1 - \ln \frac{\ell}{\alpha}\right).
\]

The key difference to the case of fund manager payoffs is that the critical value now depends on the nominal interest rate \( R_S \) which is endogenous and depends on \( \hat{\theta} \) itself. The two parameters now have to be determined jointly.

In the initial period \( t = 0 \) the nominal interest rate has to be set so short-term creditors receive their outside option \( R_S^* \)

\[
F(\hat{\theta}) \frac{\ell}{\alpha} + \int_0^1 \theta R_S dF(\theta) = R_S^*.
\]

Similarly for the long-term creditors

\[
\int_\hat{\theta}^1 \theta R_L dF(\theta) = R_L^*.
\]
We see that equations (4) and (5) jointly determine \( \hat{\theta} \) and \( R_S \) for any given \( \alpha \).

The firm chooses \( \alpha \) to maximize its expected profit

\[
\int_{\hat{\theta}}^{1} \theta [X - \alpha R_S - (1 - \alpha) R_L] dF(\theta)
\]

subject to equations (4), (5) and (6). This is equivalent to the problem

\[
\max_{\alpha} \left\{ F(\hat{\theta}) \ell + \int_{\hat{\theta}}^{1} \theta X dF(\theta) - \alpha R^*_S - (1 - \alpha) R^*_L \right\}
\]

s.t. \( \hat{\theta} = \left( 1 - \ln \frac{\ell}{\alpha} \right) \frac{\int_{\hat{\theta}}^{1} \theta dF(\theta)}{R^*_S - F(\hat{\theta}) \frac{\ell}{\alpha}} \)

As before, the firm maximizes the project’s economic surplus net of opportunity costs subject to an equation that implicitly defines \( \hat{\theta} \) as a function of \( \alpha \) and the exogenous parameters. The first order condition has the same structure as in the fund-manager case

\[
f(\hat{\theta}) \frac{d\hat{\theta}}{d\alpha} \left( \ell - \hat{\theta} X \right) - R^*_S + R^*_L = 0,
\]

except that the expression for \( \frac{d\hat{\theta}}{d\alpha} \) will now be different. In particular additional restrictions will be necessary to guarantee that \( \frac{d\hat{\theta}}{d\alpha} \) is positive.

**C Case \( \alpha \leq \ell \)**

This section considers the case where the mass of short-term creditors is small enough so they cannot cause the firm to fail. This corresponds to values of \( \alpha \leq \ell \), such that withdrawals from all short-term creditors can be satisfied in \( t = 1 \) without liquidating the entire project. I assume that in \( t = 2 \) short-term debt is senior to long-term debt. As in the main part I start by deriving the endogenous interest rates and the firm’s expected payoff without aggregate uncertainty.

Without liquidity risk, the expected payoff to a fund manager from rolling over is \( \theta bY \) regardless of the number of others withdrawing and the payoff to withdrawing
is $Y$ as before. The critical value for $\theta$ is therefore independent of $\alpha$ and given by

$$\tilde{\theta} = \frac{1}{\tilde{b}}.$$ 

The expected payoff to a short-term creditor is now

$$F(\tilde{\theta}) + \int_{\tilde{\theta}}^{1} \theta \tilde{R}_S dF(\theta),$$

which gives us an endogenous short-term interest rate of

$$\tilde{R}_S = \frac{R^* - F(\tilde{\theta})}{\int_{\tilde{\theta}}^{1} \theta dF(\theta)}.$$ 

The expected payoff to long-term creditors is a little more complicated. If $\alpha$ is small enough, then even after liquidation of $\alpha/\ell$ the remaining assets are large enough to pay off long-term debt:

$$\left(1 - \frac{\alpha}{\ell}\right) X > (1 - \alpha) \tilde{R}_L$$

In this case the expected payoff to long-term creditors is

$$\int_{0}^{1} \theta \tilde{R}_L dF(\theta),$$

resulting in an endogenous long-term interest rate of

$$\tilde{R}_L^{\alpha} = \frac{R^*}{\int_{0}^{1} \theta dF(\theta)}.$$ 

As $\alpha$ increases, the liquidations will start to eat into the assets available to pay off long-term debt:

$$\left(1 - \frac{\alpha}{\ell}\right) X \leq (1 - \alpha) \tilde{R}_L.$$
In this case the expected payoff to long-term creditors is

$$
\int_{0}^{\tilde{\theta}} \theta \frac{1 - \frac{\alpha}{\ell}}{1 - \alpha} X dF(\theta) + \int_{\tilde{\theta}}^{1} \theta \tilde{R}_L dF(\theta),
$$

resulting in an endogenous nominal rate of

$$
\tilde{R}_L^b = \frac{R^* - \int_{0}^{\tilde{\theta}} \theta \frac{1 - \frac{\alpha}{\ell}}{1 - \alpha} X dF(\theta)}{\int_{\tilde{\theta}}^{1} \theta dF(\theta)}.
$$

We therefore have that for small $\alpha$ the firm’s expected payoff is

$$
\int_{0}^{\tilde{\theta}} \theta \left[ (1 - \frac{\alpha}{\ell}) X - (1 - \alpha) \tilde{R}_L^a \right] dF(\theta) + \int_{\tilde{\theta}}^{1} \theta \left[ X - \alpha \tilde{R}_S - (1 - \alpha) \tilde{R}_L^a \right] dF(\theta)
= F(\tilde{\theta}) \frac{\alpha}{\ell} \ell + \int_{0}^{\tilde{\theta}} \theta \left( 1 - \frac{\alpha}{\ell} \right) X dF(\theta) + \int_{\tilde{\theta}}^{1} \theta X dF(\theta) - R^*,
$$

which is linear in $\alpha$. For larger $\alpha$ the firm’s expected payoff is

$$
\int_{0}^{1} \theta \left[ X - \alpha \tilde{R}_S - (1 - \alpha) \tilde{R}_L^a \right] dF(\theta)
= F(\tilde{\theta}) \frac{\alpha}{\ell} \ell + \int_{0}^{\tilde{\theta}} \theta \left( 1 - \frac{\alpha}{\ell} \right) X dF(\theta) + \int_{\tilde{\theta}}^{1} \theta X dF(\theta) - R^*.
$$

which is the same as in the case of small $\alpha$. Combining this with the firm’s expected payoff for $\alpha > \ell$ derived in Section 4.1 the complete expected payoff of the firm choosing $\alpha \in [0, 1]$ is

$$
\begin{cases}
F(\tilde{\theta}) \alpha + \int_{0}^{\tilde{\theta}} \theta X dF(\theta) + \int_{\tilde{\theta}}^{1} \theta X dF(\theta) - R^* & \text{for } \alpha \leq \ell, \\
F(\tilde{\theta}) \ell + \int_{\tilde{\theta}}^{1} \theta X dF(\theta) - R^* & \text{for } \alpha > \ell.
\end{cases}
$$

The payoff is continuous in $\alpha$ since the two expressions are the same for $\alpha = \ell$ but not differentiable at $\alpha = \ell$. It is either monotone or single-peaked. Due to the linearity of the firm’s expected payoff for $\alpha \leq \ell$, the optimal solution will be either $\alpha = 0$, $\alpha = \ell$, or we will be in the region $\alpha > \ell$ discussed in the main part of the paper.
To guarantee that the solution falls into the range of $\alpha > \ell$ we have to assume that the derivative of both pieces are positive at $\alpha = \ell$:

\[
\begin{cases}
F \left( \frac{1}{b} \right) - \frac{X}{\ell} \int_0^{\frac{1}{b}} \theta f (\theta) d\theta > 0 \\
f \left( \frac{1}{b} \right) \frac{1}{\ell b} (\ell - \frac{1}{b} X) > 0
\end{cases}
\]

These conditions involve only exogenous parameters and can be satisfied.

With aggregate uncertainty, the firm’s expected payoff is more complicated

\[
\begin{cases}
p \left[ F(\hat{\theta}) + \int_{\frac{1}{bL}}^{\hat{\theta}} \left( 1 - \frac{\theta}{\ell L} \right) \theta X dF(\theta) + \int_{\hat{\theta}}^{\frac{1}{bL}} \theta X dF(\theta) - R^* \right] \\
\quad + (1 - p) \left[ F(\hat{\theta}) + \int_{\frac{1}{bL}}^{\hat{\theta}} \left( 1 - \frac{\theta}{\ell L} \right) \theta X dF(\theta) + \int_{\hat{\theta}}^{\frac{1}{bL}} \theta X dF(\theta) - R^* \right] \quad \text{for } \alpha < \ell_L
\end{cases}
\]

\[
\begin{cases}
p \left[ F(\hat{\theta}) + \int_{\frac{1}{bL}}^{\hat{\theta}} \left( 1 - \frac{\theta}{\ell L} \right) \theta X dF(\theta) + \int_{\hat{\theta}}^{\frac{1}{bL}} \theta X dF(\theta) - R^* \right] \\
\quad + (1 - p) \left[ F(\hat{\theta} L) + \int_{\frac{1}{bL}}^{\hat{\theta} L} \theta X dF(\theta) - R^* \right] \quad \text{for } \ell_L < \alpha < \ell_H
\end{cases}
\]

\[
\begin{cases}
p \left[ F(\hat{\theta} H) + \int_{\frac{1}{bH}}^{\hat{\theta} H} \theta X dF(\theta) - R^* \right] \\
\quad + (1 - p) \left[ F(\hat{\theta} L) + \int_{\frac{1}{bL}}^{\hat{\theta} L} \theta X dF(\theta) - R^* \right] \quad \text{for } \alpha > \ell_H
\end{cases}
\]

Again these are conditions involving only exogenous parameters and can be satisfied.
References


