1 Supplementary Notes

Consider the following PDE:

\[ \frac{\partial u}{\partial t} + L(u) = S(x,t) \]  \hspace{1cm} (1)

where \( S(x,t) \) is a “source” term and \( L \) is the following linear differential operator:

\[ L(u) \equiv -\frac{\partial^2}{\partial x^2}. \]  \hspace{1cm} (2)

The domain of interest is \( -\ell \leq x \leq \ell \) and \( t \geq 0 \). The initial condition is:

\[ u(x,0) = f(x). \]  \hspace{1cm} (3)

The boundary conditions are:

\[ u(-\ell, t) = L(t), \] \hspace{1cm} (4a)
\[ u(\ell, t) = R(t). \] \hspace{1cm} (4b)

The functions \( S(x,t), f(x), L(t) \) and \( R(t) \) are piecewise continuous functions, but are otherwise arbitrary. The following is easily verified (using integration by parts twice):

\[ < L(u), v > = < u, L(v) > + \left[ u \left( \frac{\partial v}{\partial x} \right) - \left( \frac{\partial u}{\partial x} \right) v \right]_{x=-\ell}^{x=\ell} \]  \hspace{1cm} (5)
provided both \( u(x) \) and \( v(x) \) are twice differentiable (with respect to \( x \)) continuous functions. The term in the square bracket is called the bilinear concommitant. It vanishes if \( u(x) \) and \( v(x) \) are both zeros at the two boundary points. In the general case—when \( u(x) \) and \( v(x) \) are not required to be zeros there—the bilinear concommitant must remain there.

### 1.1 The Critical Mass Problem of the Atomic Bomb

Consider the case \( S = \gamma u \) (\( \gamma \) is a constant), \( L = R = 0 \). You may be interested to know that this PDE with \( \gamma > 0 \) is the governing equation relevant to the ignition problem of the fission atomic bomb—\( u \) is related to the neutron density of the fissionable rod which is located in \(-\ell \leq x \leq \ell\). The constant \( \gamma \) has the dimension of reciprocal time; \( \gamma u \) represents the spontaneous fission activities of the fissionable material. At the two ends the rod is capped by material that absorbs neutrons (such as lead) so the \( L = R = 0 \).

For this problem, \( u = 0 \) is an exact solution when the initial condition is \( u(x,0) = 0 \). Is this (nothing is happening) solution unique?

Let us do what we did in Notes 3a to investigate uniqueness. Multiplying by \( u \) and integrating over \( x \), we have:

\[
\frac{d}{dt} \left( \int_{-\ell}^{\ell} \frac{u^2}{2} \, dx \right) = + \int_{-\ell}^{\ell} \left\{ - \left( \frac{\partial u}{\partial x} \right)^2 + \gamma \frac{u^2}{2} \right\} \, dx \tag{6}
\]

If \( \gamma \leq 0 \), we can conclude, just like what was done before, that the trivial solution \( u = 0 \) is unique. If initially \( u(x,0) \) is not precisely zero but is small, the integral of \( u^2/2 \) will get smaller and smaller as time proceeds.

However, if \( \gamma > 0 \), the previous arguments for uniqueness can fail. In fact, we can show that \( u \) can “explode” if \( u(x,0) \) is not precisely zero!

Consider a separation of variable solution for \( u \):

\[
u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) \tag{7} \]

Substituting this into the PDE, and setting the \( n \)-th term to zero, we obtain:

\[
\frac{1}{T_n} \frac{dT_n}{dt} - \frac{1}{X_n} \frac{d^2X_n}{dx^2} = \gamma. \tag{8}
\]
Going through the standard arguments, we choose as our “separation constant” $\mu_n^2$ so that:

$$\frac{d^2X_n}{dx^2} + \mu_n^2X_n = 0.$$  

(9)

The “even” solution is:

$$X_n = \cos(\mu_n x).$$

(10)

Honoring the boundary conditions $X(-\ell) = X_n(\ell) = 0$, we determine the separation constant $\mu_n$:

$$\mu_n = \frac{2n + 1}{2\ell} \pi, \quad n = 0, 1, 2, \ldots, \infty.$$  

(11)

Using (9) and this $\mu_n$ into (8), we obtain:

$$\frac{dT_n}{dt} + \sigma_n T_n = 0$$  

(12)

with solution

$$T_n = C_n \exp(-\sigma_n t)$$  

(13)

where $C_n$ is an integration constant and

$$\sigma_n = -\gamma + \left(\frac{(2n + 1)\pi}{2\ell}\right)^2.$$  

(14)

Inspection of (13) shows that $T_n$ decays with time only when $\sigma_n$ is positive. In other words, $u(x, t)$ will explode if any of the $\sigma_n$’s is negative. Since the smallest possible $\sigma_n$ is $\sigma_0$, we conclude that the bomb explodes if $\sigma_0$ is negative. The critical mass condition is thus:

$$\ell^2 \gamma \geq \left(\frac{\pi}{2}\right)^2.$$  

(15)

For a given fissionable material, $\gamma$ is fixed. It is clear that a sufficiently short rod is “stable.” To make a bomb, you need to have $\ell$ long enough so that the above condition is satisfied (you can put two subcritical rods together to make the longer rod supercritical!). Then, POW!

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1Actually, you need to check the Sine solution for $X_n$ to see whether it reaches criticality first or not. It is readily verified that it does not.
1.2 Green’s Function

The method of separation of variables involves a prodding, mechanical procedure. You plow through the messy algebra and computation, and then you plot your results and look at the graphs.

There is a much prettier way to go.

Consider now a function $G(x, t; x_o, t_o)$ in our domain of interest where $x_o$ and $t_o$ are parameters with $t_o > 0$ and $-\ell \leq x_o \leq \ell$. This $G(x, t; x_o, t_o)$, called the Green’s function, is defined to be a solution of the following PDE:

$$\frac{-\partial G}{\partial t} + L(G) = \delta(t - t_o)\delta(x - x_o) \quad (16)$$

where $\delta(\xi)$ denotes a Dirac Delta function located at $\xi = 0$ (I hope you remember what a Dirac Delta function is). Note that the “future” of (16) is in the direction of $-t$. We impose the “initial condition” that $G = 0$ for all $x$ in the “past,” i.e. $t > t_o$. We further assume that $G = 0$ at $|x| = \pm \ell$.

Hence, the “reason” $G$ is not zero everywhere is because of the presence of the right hand side of (16). It is most important to realize that $G = 0$ for $t > t_o^+$—because the right hand side hasn’t “done anything” yet. For the moment, let us assume that somehow we have found $G(x, t; x_o, t_o)$.

Taking the inner product of (1) with $G$, and subtracting from it the inner product of $u$ with (16), we have:

$$d < u, G > \frac{dt}{dt} + \left[u \left(\frac{\partial G}{\partial x}\right)\right]^{x=\ell}_{x=-\ell} = -\delta(t - t_o) < u, \delta(x - x_o) > + < S, G >$$

$$= -\delta(t - t_o)u(x_o, t)+ < S, G > . \quad (17)$$

The first term on the left hand side is written as $d/dt$ because $< u, G >$ has no $x$-dependence; the second term is the bilinear concomitant except that one of the terms vanished because $G$ is zero at both boundary points.

Let us now integrate (17) over our time domain of interest:

$$t \geq 0. \quad (18)$$

Since $G = 0$ for $t > t_o^+$, the actual integration needs only to be perform for $0 \leq t \leq t_o^+$. We obtain:

$$u(x_o, t_o) = -\int_{0}^{t_o^+} \left\{ \frac{d < u, G >}{dt} + \left[u \left(\frac{\partial G}{\partial x}\right)\right]^{x=\ell}_{x=-\ell} - < S, G > \right\} dt \quad (19)$$
\[ u(x_o, t_o) = \int_{-\ell}^{\ell} f(x)G(x, 0; x_o, t_o)dx + \int_{0}^{t_o} \left( \int_{-\ell}^{\ell} S(x, t)G(x, t; x_o, t_o)dx \right)dt \]

\[ - \int_{0}^{t_o} R(t)\frac{\partial G(\ell, t; x_o, t_o)}{\partial x}dt + \int_{0}^{t_o} L(t)\frac{\partial G(-\ell, t; x_o, t_o)}{\partial x}dt. \]

The integration of the first term took advantage of the fact that \( G(x, t_o^+; x_o, t_o) = 0 \). To write the whole thing out long hand, we have:

Everything on the right hand side of this equation is known; the first term represents the effect of initial condition, the second term represents the effect of the source term, the third term represents the effect of the right boundary condition, and the last term represents the effect of the left boundary conditions. The punch line is: if \( G(x, t; x_o, t_o) \) is indeed known, then the solution for \( u(x, t) \) for any initial condition \( f(x) \) is explicitly given, and is reduced to a matter of “quadrature.”

### 1.3 Finding the Green’s Function for \( \ell = \infty \)

It is a straightforward matter to find \( G_\infty(\xi, \tau) \) by Fourier Transform (entry #6 of Appendix D shall be useful in this effort)—since the Fourier Transform of Dirac Delta function is one of the easiest thing to do in the Fourier Transform world. But here we will do it a different way.

Let us introduce new independent variables:

\[ \xi = x - x_o, \]
\[ \tau = t_o - t. \]

We denote the Green’s function for the unbounded \( x \)-domain by \( G_\infty \). The governing equation (see (16)) is now:

\[ \frac{\partial G_\infty}{\partial \tau} - \frac{\partial^2 G_\infty}{\partial \xi^2} = \delta(\xi)\delta(\tau). \]
Hence, we have $G_\infty = G_\infty(\xi, \tau)$. The boundary condition is $G_\infty(\pm\infty, \tau) = 0$ for all $\tau$. The initial condition is $G_\infty(\xi, 0^-) = 0$ for all $\xi$. Alternatively, we can say that the initial condition is:

$$G_\infty(\xi, 0^+) = \delta(\xi). \tag{24}$$

Hope you have no difficulty in deriving the above.

For $\xi > 0^+$, the right hand side of (23) is zero. Integrating (23) with respect to $x$ from $-\infty$ to $+\infty$, we obtain:

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} G_\infty dx = 0, \tag{25}$$

or,

$$Q = \int_{-\infty}^{\infty} G_\infty dx = \text{constant}. \tag{26}$$

In other words, $Q$, the total “area” under the $G_\infty$ curve, does not change with time.

Normally, $\xi$ has the dimension of length. It is clear from (23) that $\tau$ here (after absorbing the dimensional $\alpha^2$) has the dimension of length-squared. Here is a simple-minded question: suppose you want to non-dimensionalize $\xi$, what length would you use?

An intelligent answer is, let’s use square root of $\tau$. We now introduce a new “similarity variable $\eta$ to replace $\xi$:

$$\eta = \frac{\xi}{\sqrt{\tau}}. \tag{27}$$

Let us now try the following (brilliant) attempt at separation of variables:

$$G_\infty(\xi, \tau) = T(\tau)X(\eta) \tag{28}$$

In order to satisfy (25), we must require:

$$T(\tau) \propto \frac{1}{\sqrt{\tau}}. \tag{29}$$

Substituting (28) and (29) into (23) and limiting our attention to $\tau \geq 0^+$, we obtain:

$$\frac{d^2X}{d\eta^2} + \frac{1}{2} \frac{d(\eta X)}{d\eta} = 0 \tag{30}$$
Integrating, we have:
\[
\frac{dX}{d\eta} + \frac{\eta X}{2} = 0. \tag{31}
\]
Integrating once more, we have:
\[
X = \exp(-\frac{\eta^2}{4}). \tag{32}
\]
Putting all the pieces together (including (25)), we have:
\[
G_\infty(x, t; x_o, t_o) = Q \frac{2}{\sqrt{\pi t}} \exp\left(-\frac{(x-x_o)^2}{4(t_o-t)}\right) \tag{33}
\]
\(G_\infty(x, t; x_o, t_o)\) is usually called the *fundamental Green’s function* of the one-dimensional diffusion PDE. If our domain of interest is all values of \(x\), this is the Green’s function of choice.

### 1.4 Method of Images

Consider the domain of interest \(0 \leq x \leq \infty\) and \(t \geq 0\). Let’s denote this Green’s function by \(G_+(x, t; x_o, t_o)\).

- We would like \(G_+(x, t; x_o, t_o)\) to satisfy (23), but vanishes on \(x = 0\).

  Consider the following guess:

  \[
  G_+(x, t; x_o, t_o) = G_\infty(x, t; x_o, t_o) - G_\infty(x, t; -x_o, t_o) \tag{34}
  \]

  What is this concoction? You can readily verified that it satisfies (23) in the current domain of interest. Most importantly, it is readily verified that \(G_+(0, t; x_o, t_o) = 0\).

- We would like \(G_+(x, t; x_o, t_o)\) to satisfy (23), but \(\partial G_+ / \partial x\) vanishes on \(x = 0\).

  Consider the following guess:

  \[
  G_+(x, t; x_o, t_o) = G_\infty(x, t; x_o, t_o) + G_\infty(x, t; -x_o, t_o) \tag{35}
  \]

  What is this concoction? You can readily verified that it satisfies (23) in the current domain of interest. Most importantly, it is readily verified that \(\partial G_+ / \partial x\) vanishes on \(x = 0\).

What we have done is to put an “image” of the Dirac Delta Function on the other side of \(x = 0\) with the appropriate sign. Very clever!