Continuum Theory of Spherical Electrostatic Probes

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A continuum theory for spherical electrostatic probes in a slightly ionized plasma is developed. The density of the plasma is taken to be sufficiently high such that both ions and electrons suffer numerous collisions with the neutrals before being collected by an absorbing probe. A general discussion of probes at an arbitrary potential is given. It is found that for very negative probe potentials the sheath thickness can be comparable to the probe radius. Two explicit forms of current–voltage characteristics are given; one for very negative probes, the other for probes at nearly plasma potential.

Both of these are based on the assumption that the probe radius is large compared with the Debye length. Numerical computation is also given for negative probes of a wider range of probe sizes.

1. INTRODUCTION AND FORMULATION

Most investigations of electrostatic probes in the literature deal with low-density plasmas. In these analyses the probe dimension was assumed to be small compared with the mean free path of the ions such that on the way to the collecting surface, the ions experienced no collision with any other particles. The difficulties encountered in using a low-density theory for a discharge at pressure of about 1 mm Hg were pointed out by Boyd. The main objection is that the probe will in general be too small to operate satisfactorily. In the same paper, Boyd set up a theoretical model aiming at the formulation of a probe theory for high-density discharge. Earlier, in 1936, Davydov and Zmanovskaja also considered the same problem. Both Davydov and Zmanovskaja and Boyd have found it necessary to assume a priori a sheath region; the latter even concluded that the problem is solvable only when the sheath thickness is first determined by experiment. Their models were crude so that their results are open to severe criticisms. In the present study, we shall consider a negative probe in a high-density, slightly ionized quiescent plasma. The logical starting point will be a set of hydrodynamic equations for a mixture and a set of diffusion flux equations deduced from the principle of linear irreversible thermodynamics. We assume that the plasma is only slightly ionized, and the bulk of the plasma is quiescent. Since the mass velocity of the plasma will be small compared with that due to diffusion, the equation of motion, therefore, does not enter into consideration. The equations of continuity for ions and electrons are

\[ \frac{\partial N_i}{\partial t} + \nabla \cdot \Gamma_i = 0, \]

\[ \frac{\partial N_e}{\partial t} + \nabla \cdot \Gamma_e = 0, \]

where \( N_i, N_e \) are number densities of ions and electrons and the \( \Gamma \)'s are number flux densities given by

\[ \Gamma_i = N_i \mathbf{v}_i = -D_i \nabla N_i - \langle D_i N_i eZ/kT \rangle \nabla \phi, \]

\[ \Gamma_e = N_e \mathbf{v}_e = -D_e \nabla N_e + \langle D_e N_e e/kT_e \rangle \nabla \phi, \]

where the \( D \)'s are diffusion coefficients, and \( \phi \) is the electrostatic potential. Other notations are standard ones.

The potential distribution is determined by Poisson's equation

\[ \nabla^2 \phi = -4\pi e(N_i Z - N_e). \]

For steady state and spherical geometry (1.1) and (1.2) can be integrated once easily. We designate the integration constants by \( I \) and \( I_e \), which are the total ion and electron current, respectively. Thus

\[ 4\pi r^2 Z e D_i \left( \frac{dN_i}{dr} + \frac{N_i eZ \phi}{kT} \frac{d\phi}{dr} \right) = I, \]

\[ 4\pi r^2 e D_e \left( \frac{dN_e}{dr} - \frac{N_e e \phi}{kT_e} \frac{d\phi}{dr} \right) = I_e. \]

Equation (1.3) is then

\[ \frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = -4\pi e(N_i Z - N_e), \]

where \( r \) is the radial coordinate.
The boundary conditions are

\[ \phi(r_p) = \phi_v, \quad \phi(\infty) = 0, \quad (1.7) \]

where \( r_p \) is the probe radius and the subscript \( p \) will always indicate conditions on the probe surface. We introduce nondimensional variables \( x, y, n, n_e \) as

\[ x = \frac{h_e}{a} \frac{1}{r} \quad \text{with} \quad h_e^2 = \frac{k T_e}{4 \pi N_e e^2}, \quad (1.8) \]

\[ y = -\frac{e \phi}{k T_e}, \quad (1.9) \]

\[ n = \frac{Z N_l}{N_0}, \quad n_e = \frac{N_e}{N_0}. \quad (1.10) \]

The dimensionless quantity \( a \) is introduced to put (1.4), (1.5), (1.6), into a simpler form, i.e.,

\[ \frac{d n}{dx} + \frac{d y}{dx} n_e = -\mu, \quad (1.11) \]

\[ \frac{d n}{dx} - \frac{d y}{dx} n = -1, \quad (1.12) \]

\[ a^2 x^4 \frac{d^2 y}{dx^2} = n - n_e, \quad (1.13) \]

where

\[ \epsilon = \frac{T}{Z T_e}, \quad (1.14) \]

\[ a = \frac{4 \pi N_e e D_i h_e}{\epsilon l}, \quad (1.15) \]

\[ \mu = \frac{I_{\mu_1}}{I_{\mu_e}}, \quad (1.16) \]

and \( \mu_1, \mu_e \) are mobilities of ions and electrons, respectively. In arriving at (1.16), Einstein's relations

\[ \mu_i = \frac{e D_i Z}{k T}, \quad \mu_e = \frac{e D_e}{k T_e}, \]

have been used. The corresponding boundary conditions for Eqs. (1.11), (1.12), (1.13) are

\[ y(x_p) = y_v, \quad y(0) = 0, \quad (1.17) \]

\[ n(x_p) = n_e(x_p) = 0, \quad n(0) = n_e(0) = 1. \quad (1.18) \]

Formally, our mathematical problem is to find the following functions:

\[ y = y(x; y_v, \rho_v, \epsilon), \quad (1.19a) \]

\[ n = n(x; y_v, \rho_v, \epsilon), \quad (1.19b) \]

\[ n_e = n_e(x; y_v, \rho_v, \epsilon), \quad (1.19c) \]

\[ a = a(y_v, \rho_v, \epsilon), \quad (1.19d) \]

\[ \mu = \mu(y_v, \rho_v, \epsilon), \quad (1.19e) \]

where \( \rho_v = r_p/h_e \) and is a dimensionless parameter specifying the size of the probe. Note that the value of \( x_p \) can be obtained from (1.19a) implicitly as the solution of

\[ y_p = y(x_p; y_v, \rho_v, \epsilon), \quad (1.19f) \]

or explicitly from (1.8) and (1.19d) as

\[ x_p = \frac{1}{\rho_v} \frac{1}{a(y_v, \rho_v, \epsilon)} \]

\[ = x_v(y_v, \rho_v, \epsilon). \quad (1.19g) \]

It is seen that a set of solutions is specified by three independent parameters, \( \rho_v, y_v, \) and \( \epsilon. \)

One particular solution of (1.11), (1.12), (1.13), (1.17), (1.18) is readily found to be

\[ y(x) = 0, \quad (1.20) \]

\[ n(x) = n_e(x) = 1 - x/\epsilon, \]

with

\[ \mu = \frac{1}{\epsilon}, \quad (1.21) \]

\[ a = \frac{1}{\rho_v} \frac{h_e}{h_v} = \frac{\mu}{\rho_v}. \quad (1.22) \]

This corresponds to the case when the probe is at the plasma potential, which is taken to be zero in the present analysis. Using (1.15) and (1.22) we find for this special case the ion current to be

\[ I = I_R = 4 \pi N_e e D_i v_{th}. \quad (1.23) \]

It is interesting to note that \( I_R \) is proportional to \( r_p. \)

The corresponding ion number flux per unit probe area is

\[ \Gamma_R = \frac{I_R}{4 \pi r_p^2 Z} = \frac{N_e v_{th}}{4 \sqrt{3}} \left( \frac{1}{r_p} \right), \quad (1.24) \]

where \( D_i \) is taken to be \( \frac{1}{2} v_{th} \) and \( l, v_{th} \) are the mean free path and the thermal speed of ions, respectively. Equation (1.24) gives the random ion number flux which agrees with formula (11), page 34 of reference 7. Note it is reduced by a factor \( \frac{3}{4} (l/r_p) \) compared with the case when collisions are absent.

At this point, we can give a clear physical interpretation for \( x_p. \) Using (1.23), it is seen that (1.15) can be written as

\[ \frac{d y}{d x} = \frac{1}{\rho_v} \frac{1}{a(y_v, \rho_v, \epsilon)} \]

\[ = \frac{1}{\rho_v} \frac{h_e}{h_v} = \frac{\mu}{\rho_v}. \quad (1.22) \]


\[ \text{For an absorbing surface, the values of } N_I \text{ and } N_e \text{ on the probe surface are of order of mean free path (between charged and neutral particles) divided by a characteristic length. For the latter, a factor of } T_e/T \text{ is needed. Within the continuum formulation they are taken to be zero.} \]
\[ a = \frac{I_0}{I} \frac{1}{\varepsilon \rho_0}. \] 

(1.25)

Using the first expression in (1.8), we have

\[ x_\nu = \varepsilon (I/I_0). \] 

(1.26)

Therefore, the value of \( x_\nu \) is proportional to the ion current collected by the probe. The relation (1.19g) is then explicitly the ion current–voltage characteristic of the probe. One of the objectives of the present theory is to construct this characteristic.

The governing equations (1.11), (1.12), and (1.13) are highly nonlinear. The boundary conditions (1.17) and (1.18) are specified at two points \( x = 0 \) and \( x = x_\nu \) with the position of the latter unknown. The two integration constants, \( \mu \), and \( a \), are also unknown and must be found in the process of solution. Furthermore, as was mentioned previously, a set of solutions is specified by three independent physical parameters, \( y_\nu \), \( \rho_\nu \), and \( \varepsilon \). Although the governing equations are ordinary differential equations, the solutions are difficult to construct even with the aid of modern high-speed digital computers. In the present paper, we study several interesting limiting cases and take advantage of simplifying approximations that are available.

In Sec. II, the orders of magnitude of \( a \) and \( \mu \) are estimated. It is found that \( \mu \) will be a small number provided that \( y_\nu \) is sufficiently large. Also, \( a \) will be a small number provided that \( \rho_\nu \) is sufficiently large. Taking advantages of these observations, attention is confined to large probes (\( a \ll 1 \)) and highly negative probe potentials (\( y_\nu \gg 1 \)). The Boltzmann electron density distribution is shown to be a valid approximation in the limit of \( \mu \to 0 \). In Sec. III, detailed analysis is carried out to study the limiting solutions for \( a \ll 1 \) and \( y_\nu \gg (2/3) \ln (1/a) \). Within the framework of this analysis, it is found that the solution \( y = y(x; \ y_\nu, \ \rho_\nu, \ \varepsilon) \) can be written as

\[ y = y(\bar{x}; \ \bar{a}, \ \varepsilon), \]

and similarly for \( n \) and \( n_\nu \), where \( \bar{x} = x(1 + \varepsilon)^{-1} \), \( \bar{a} = a(1 + \varepsilon)^{1/4} \). The value of \( \bar{x}_\nu \), which is still proportional to the ion current, can be written as

\[ \bar{x}_\nu = \frac{1}{\partial \bar{x}_\nu} = \bar{x}_\nu(y_\nu, \ \bar{\rho}_\nu, \ \varepsilon), \]

(1.28)

where \( \bar{\rho}_\nu = \rho_\nu(1 + \varepsilon)^{1/4} \). Comparing (1.27) to (1.19a), it is seen that the number of independent parameters have been reduced by one and therefore the problem is greatly simplified. Consistent with the approximations introduced, it is found that the solution is in general divided into four distinct regions in each of which different approximations are available. The plasma is almost neutral far away from the probe, and the solution is represented by the quasi-neutral solution. Since the probe potential is highly negative near the probe, the electron density is negligible compared with the ion density, and the solution is represented by the ion-sheath solution. A transitional region exists between the quasi-neutral region and the ion-sheath region and its structure is carefully analyzed. Immediately adjacent to the probe, diffusion of ions dominates, and the solution there is represented by the ion-diffusion solution in the form of a boundary layer. The effects of \( \varepsilon \) on the solutions are carefully analyzed, and are found to be important only in determining the detailed structures inside the transitional and ion-diffusion regions, and are relatively unimportant in the determination of the current. In addition, the boundary condition for the ion density at the probe is found to be irrelevant.

In Sec. IV, numerical computations are carried out to supplement the analysis of Sec. III. The values of \( a \) need not be small in principle; although for practical reasons, only small and moderate values are considered. In Sec. V, two different current–voltage characteristics are constructed by cross-plotting the numerical solutions. From the results of Sec. III, the ion current–voltage characteristic is also constructed analytically.

In Sec. VI, a discussion of probes at an arbitrary potential is given. It is pointed out there that very negative probes are quite distinct from the probe of moderate probe potential. An analysis of the probes near the plasma potential is also given. To contrast these large \( \rho_\nu \gg 1 \) solutions, the limiting solution for \( \rho_\nu \ll 1 \) is presented in Appendix I.

In Sec. VII, the basic continuum assumption underlying the present theory is discussed. It is found that when \( \rho_\nu \gg 1 \), the main disturbances in the plasma are confined to a layer adjacent to the probe surface. For a probe at nearly plasma potential, the thickness of this layer is of the order \( r_se^4/\rho_\nu(1 + \varepsilon)^{1/4} \). Thus formally the continuum assumption is valid, provided that

\[ \frac{r_se^4}{\rho_\nu(1 + \varepsilon)^{1/4}} \gg l \quad \text{or} \quad (1 + \varepsilon)^{1/4}\left(\frac{l_{\ast}}{h_1}\right)^{1/4} \ll 1, \]

where \( l \) is the characteristic mean free path, and \( h_1 \) is the ion Debye length. On the other hand, for probes at very highly negative potential, the thickness of the sheath divided by the probe radius is roughly proportional to the square root of the probe potential. The continuum assumption will never
present any problem in very high probe potential cases. When \( p_o \ll 1 \), the criterion for the validity of theory is \( l/r_o \ll 1 \).

II. ESTIMATIONS OF \( a \) AND \( y \)

Equation (1.25) can readily be used to estimate the order of magnitude of \( a \). For probe potentials near the plasma potential, \( I/I_n \) will be order one, and \( a \) is of the order \( h_e/(e r_p) \). Thus, \( a \) will be a small number if \( h_e/r_o \) is small, provided that \( e \) is not close to zero. As the probe potential \( \phi_p \) increases negatively, the total ion current must also increase. For high \( y_o \), as we shall see later,

\[
a \sim \left( h_e/r_o y_o \right)^{\frac{1}{2}}. \tag{2.1}
\]

Hence from (2.1), \( a \) will be a small number if \( y_o \gg 1 \). When \( a^2 \) is small, the left-hand side of (1.13) can be dropped in a region extending from \( x = 0 \), so long as \( x^2 \) remains of order one. This region is called the plasma or quasi-neutral region. Solving for \( n (= n_e) \) and \( y(x) \) from (1.11) and (1.12), we obtain the following solutions:

\[
n(x) = n_e(x) = 1 - \frac{1 + \mu}{1 + \epsilon} x + O(a^2), \tag{2.2}
\]

\[
y(x) = -\frac{1 - \mu \epsilon}{1 + \mu} \ln \left( 1 - \frac{1 + \mu}{1 + \epsilon} x \right) + O(a^2). \tag{2.3}
\]

Equation (2.3) is singular at \( x = x_s = (1 + \epsilon)/(1 + \mu) \). Using (2.2) and (2.3), it is readily shown that \( a^2 x^4 y dx \) becomes comparable either with \( n \) or \( n_e \) when

\[
1 - \frac{x}{x_s} = \frac{a^2 (1 + \epsilon)^{\frac{1}{2}} (1 - \epsilon \mu)^{\frac{1}{2}}}{1 + \mu}. \tag{2.4}
\]

Thus the quasi-neutral solutions (2.2), (2.3) are valid only for \( 0 \leq x < x_s \). The singular point \( x_s \) as it stands depends both on \( \epsilon \) and \( \mu \). The former is simply a constant (dependent only on plasma properties). However, \( \mu \) is unknown and depends on \( x \) and the boundary condition of electron density at the probe. From (1.11) and (1.18), the electron density is given exactly by

\[
n_e(x) = \exp (-y) \left[ 1 - \mu \int_0^x \exp(y) \, dx \right], \tag{2.5}
\]

where

\[
\mu = \left[ \int_0^x \exp(y) \, dx \right]^{-1}. \tag{2.6}
\]

The Boltzmann distribution for electrons is recovered if we set \( \mu = 0 \) in (2.5), giving

\[
n_e(x) = \exp [-y(x)]. \tag{2.7}
\]

It is seen from (2.5) that no matter how small \( \mu \) may be, (2.7) cannot be a valid approximation to (2.5) near the probe where the two terms inside the bracket of (2.5) are of the same order of magnitude. However, this will not affect the usefulness of (2.7) for highly negative probe potentials, since \( n_e \) given by (2.7) at the probe is practically zero anyway.

We shall proceed from here under the assumption that \( a \ll 1 \) and \( y_o \gg 1 \) so that both \( a \) and \( \mu \) are small quantities. No restriction shall be placed on \( e \) but it will be tacitly assumed that it is of order unity or smaller.

III. HIGHLY NEGATIVE PROBES

In this section, we consider the cases where the probe potential is sufficiently high such that Boltzmann’s distribution (2.7) is valid at least away from the probe. As a check, the values of \( \mu \) will be evaluated by (2.6) after the solution is obtained.

Eliminating \( n_e \) and \( n \) between (2.7), (1.12), and (1.13) one obtains

\[
a^2 x^4 y_{xx} = \frac{1 + \epsilon(a^2 x^4 y_{xx})_x}{y_x} - (1 + \epsilon) \exp (-y). \tag{3.1}
\]

We shall absorb the constant \( 1 + \epsilon \) in the following way:

\[
\bar{x} = x/(1 + \epsilon), \tag{3.2}
\]

\[
\bar{a} = a(1 + \epsilon)^{\frac{1}{2}}, \tag{3.3}
\]

then

\[
\bar{a}^2 \bar{x}^4 \bar{y}_{xx} = \frac{1}{y_x} \left[ 1 + \epsilon a^2 (\bar{x}^4 y_{xx})_x \right] - \exp (-y). \tag{3.4}
\]

The quasi-neutral solution (2.2), (2.3) becomes

\[
n(\bar{x}) = n_e(\bar{x}) = 1 - \bar{x} + O(a^2), \tag{3.5}
\]

\[
y(\bar{x}) = -\ln (1 - \bar{x}) + O(a^2). \tag{3.6}
\]

The singular point is now at \( \bar{x} = 1 \) (since \( \mu \equiv 0 \)). We shall proceed to construct approximate solutions \( y = y(x; a, \epsilon, y_o) \) to (3.4) satisfying appropriate boundary conditions under the assumption that \( \bar{a} \ll 1 \) and \( y_o \gg 1 \). Note that \( \bar{a} \) instead of \( r_p/h_e \) is used here as a parameter. From (2.4), it is seen that the quasi-neutral solutions (3.5) and (3.6) are valid in the range

\[
0 \leq \bar{x} < 1 - O(\bar{a}^4).
\]

The highest value of \( y \) attained by (3.6) is therefore

\[
y < \frac{1}{\mu} \ln (1/\bar{a}).
\]

We define a highly negative probe as

\[
y_o > \frac{1}{\mu} \ln (1/\bar{a}) \gg 1,
\]
so that $\tilde{z}_p > 1$. To extend the solution past $\tilde{z} = 1$, we stretch the coordinate as

$$\tilde{z} = 1 + \tilde{a}^t t,$$  \hspace{1cm} (3.7)

$$y = \frac{3}{2} \ln \left(1/\tilde{a}\right) + Y(t).$$  \hspace{1cm} (3.8)

Equation (3.4) then becomes

$$Y_{tt} = \left[(1 + \epsilon Y_{tt})/Y_t\right] - \exp \left(-Y\right) + O(\tilde{a}^4).$$  \hspace{1cm} (3.9)

To the order indicated, the equation is independent of $a$. However, it is still dependent on $\epsilon$. In what follows we investigate the effect of $\epsilon$ on $Y(t)$. [Note that in (3.9) $\epsilon$ is associated with the highest-order term $Y_{tt}$ only.]

The first integral of (3.9) which joins (3.6) smoothly at $t \to \infty$ is

$$\frac{1}{2} Y_t^2 = t + \exp \left(-Y\right) + \epsilon Y_{tt}. \hspace{1cm} (3.10)$$

Equation (3.10) needs two boundary conditions. To match the quasi-neutral solution as $t \to -\infty$, we impose

$$Y(t) = -\ln \left(-t\right) \text{ as } t \to -\infty. \hspace{1cm} (3.11)$$

For the other end, we have from (3.10)

$$\frac{1}{2} Y_t^2 = t \text{ as } t \to +\infty, \hspace{1cm} (3.12)$$

or

$$Y(t) \to (2\sqrt{2}/3)t^4 + C(\epsilon) \text{ as } t \to \infty, \hspace{1cm} (3.13)$$

where $C(\epsilon)$ is an integration constant to be determined. We shall call this region the transitional region. If $y_a \approx \frac{3}{2} \ln \left(1/\tilde{a}\right)$, the solution $Y(t)$ will terminate within the transitional region. For this case, one must consider $\mu \neq 0$ and apply the ion and electron density boundary conditions at the probe to $Y(t)$. (See Sec. VI.) In the present section we consider highly negative probes only.

Letting $z = \exp \left[-Y(t)\right]$, we write (3.10) as

$$\epsilon z_{tt} = z(t + z) - \left(\frac{1}{2} - \epsilon\right)(z_t^2/z) \hspace{1cm} (3.14)$$

and the boundary conditions (3.11) and (3.12) become

$$z(-\infty) = -t, \hspace{1cm} z(+\infty) \to 0. \hspace{1cm} (3.15)$$

The analytical solution of (3.14), (3.15) is not known, and the full numerical analysis is rather complicated. However, the detailed solution $Y(t)$ is really of interest here. The important quantity to be evaluated is the integration constant $C(\epsilon)$ in (3.13). Fortunately, for two special cases $\epsilon = \frac{1}{2}$ and $\epsilon = 0$ the solutions are fairly easy to obtain. These two special solutions provide some indication on the $\epsilon$ dependence of $C$. For $\epsilon = \frac{1}{2}$, (3.14) reduces to

$$\epsilon z_t = 2z(z + t). \hspace{1cm} (3.16)$$

The solution $z(t; \epsilon = \frac{1}{2})$ can be shown to be symmetric about $z = (\tan 3\pi/8)t$. Taking advantage of this property, the boundary condition can be written as

$$z(0) = -\frac{1}{2}, \hspace{1cm} z(\infty) = 0. \hspace{1cm} (3.17)$$

Equation (3.16) has been integrated numerically and the value of $C(\frac{1}{2})$ is found to be $3.2$. When $\epsilon = 0$, it is easier to deal directly with $Y(t)$. However, no attempt was made to integrate (3.10) for this case, since the value of $C(\epsilon = 0)$ can be obtained from later computations presented in Sec. IV. It suffices to state here that $C(0) = 3.0$. The qualitative behavior of $Y(t)$ for these two cases is sketched in Fig. 1. It is clearly seen that the effect of $\epsilon Y_{tt}$ is to shift the solution $Y(t)$ by an amount of order $\epsilon$.

Since $y_a$ is a large number, we see that if $\epsilon$ is of order unity this effect is quite negligible.

Beyond the transitional region, it is consistent to neglect $n_*$ compared with $n$. We shall call this the ion-sheath region. Dropping $\exp \left(-y\right)$ in (3.4) and letting

$$y_\pm = g(x)/\tilde{a}, \hspace{1cm} (3.18)$$

Eq. (3.4) becomes

$$x^t g_\pm = 1 + \tilde{a}e(x^2 g_\pm). \hspace{1cm} (3.19)$$
It is clear that the last term is of order \( \bar{a} \epsilon \) compared to the rest and can be neglected. Integrating (3.19), we have

\[ y^2 = 2\left(C_1 - \frac{1}{\bar{x}^3}\right) + O(\bar{a} \epsilon), \]

or

\[ y^2 = \frac{2}{3\bar{a}^2} \left(C_1 - \frac{1}{\bar{x}^3}\right), \quad (3.20) \]

where \( C_1 \) is an integration constant. \( C_1 \) will, in general, depend on \( a \). For small \( a \) when (3.12) is valid, we can match (3.20) to (3.12) as \( \bar{x} \to 1.0 \). This gives \( C_1 = 1 + O(\bar{a}^{4/3}) \). Thus

\[ y_2 = \left(\frac{2}{3}\right)^{1/2} \left(1 + 1/\bar{x}^3\right) + O(\epsilon). \quad (3.21) \]

One can integrate (3.21) once more and express the solution in terms of elliptic integrals. From (3.8) and (3.13), the boundary condition as \( \bar{x} \to 1 \) is

\[ y(\bar{x} \to 1) = \frac{2}{3} \ln (1/\bar{a}) + C(\epsilon). \quad (3.22) \]

Thus formally the solution \( y(x) \) in the ion sheath may be written as

\[
y = \left(\frac{2}{3}\right)^{1/2} \ln \left(\frac{1}{\bar{a}}\right) + C(\epsilon)
+ \left(\frac{2}{3}\right)^{1/2} \left(1/\bar{a}\right) \int_1^x \left(1 + 1/\bar{x}^3\right)^{1/2} d\bar{x}. \quad (3.23a)\]

The ion density \( n \) can be found easily and is simply

\[ n(\bar{x}) = (1 + \epsilon)/y_2. \quad (3.23b) \]

For a given \( y_p \) and \( \rho_p \), the value of \( \bar{x}_p \) can, in principle, be solved from (3.23a). The value of ion density at the probe is then given by (3.23b) evaluated at \( \bar{x}_p \). From (3.21) and (3.23b) we see that \( n(\bar{x}_p) = O(\bar{a}) \).

Next to ion sheath and adjacent to the probe surface it can be shown that there exists a thin boundary layer for the ion density distribution which we shall call the ion diffusion layer. The potential distribution within this layer is relatively unchanged from that given by (3.20). It is a straight line with the slope \( \frac{\partial}{\partial \bar{x}} = dy/d\bar{x} \) given by (3.21) evaluated at \( \bar{x} = \bar{x}_p \). The ion density distribution within this layer is given by

\[ n(\bar{x}) = \frac{1 + \epsilon}{\bar{\Lambda}} \left(1 - \exp \left[-\frac{\bar{\Lambda}}{\epsilon}(\bar{x}_p - \bar{x})\right]\right) - C_2 \exp \left[-\frac{\bar{\Lambda}}{\epsilon}(\bar{x}_p - \bar{x})\right], \quad (3.24) \]

where

\[ \bar{\Lambda} = \left(\frac{2}{3}\right)^{1/2} \left(1 + 1/\bar{x}_p^3\right) \]

and \( C_2 \) is an integration constant.

As one moves away from the probe, the exponential terms become negligible and \( n \) tends to the value given by (3.23b):

\[ n(\bar{x}) \to (1 + \epsilon)/y_2. \]

The integration constant \( C_2 \) in (3.24) is determined by the boundary value of \( n(\bar{x}) \) at the probe. It is seen that the general behavior of the full solutions is quite insensitive to the boundary conditions of ion number density at the probe. See Fig. 2.

IV. NUMERICAL INTEGRATION

The approximate solutions obtained in the previous section are useful only when \( a \) is sufficiently small. For example, in the analysis of the transitional region, terms of order \( a^3 \) are neglected compared to unity. In order to obtain solutions for moderate values of \( a \), a program of numerical computation has been carried out. The third order term involving \( \epsilon \) in (3.4) is dropped in these computations. This is a nontrivial step and the reasons for doing so are the following. First, in the light of the previous discussions, the effect of this term is well understood. Second, for \( \epsilon \) of order unity and \( y_p \gg 1 \), the error is expected to be small. Third, in most practical situations \( \epsilon \) is quite small. Finally, we shall integrate for the solution in the direction of increasing \( \bar{x} \), and this third-order term, if kept, will induce instability in the integration process. The equation is rewritten as follows:

\[
\sqrt{\frac{\bar{x}}{\bar{a}}} = \frac{1}{y_2} - \exp (-\gamma), \quad (4.1)
\]

\[ y(0) = 0, \quad y(\bar{x}_p) = y_p. \quad (4.2) \]

\* Although \( \epsilon \) is dropped from the equation, it is still retained in the transformed variables \( \bar{x}, \bar{a} \).
In the integration, the second boundary condition in (4.2) will be ignored. In place of it, \( a \) is chosen as a free parameter. The integration is started from the origin where the quasi-neutral solution (3.6) is valid.

A simple analysis will show that \( \tilde{x} = 0 \) is a singular point of (4.1); it is therefore necessary to start the actual integration at some small but finite value of \( \tilde{x} \). To obtain accurate starting values, the so-called PLK method is employed to give the first-order correction to the quasi-neutral solutions (3.6). The result is

\[
y = -\ln (1 - \eta),
\]

\[
\tilde{x} = \eta + \frac{1}{2} \tilde{a}^2 (1 - \eta)^{-2} - 4 \tilde{a}^2 (1 - \eta)^{-1} - 6 \tilde{a}^2 \ln (1 - \eta) + 4 \tilde{a}^2 (1 - \eta) + \frac{1}{2} \tilde{a}^2 (1 - \eta)^2,
\]

where \( \eta \) is a parametric variable.\(^{10}\) Pairs of values \((\eta, \tilde{x})\) are taken from (4.3), (4.4) to start the numerical integration of (4.1) using a Runge–Kutta–Gill technique, and the computed solution is considered accurate and acceptable when it is insensitive to the starting values and step size used to four significant figures. The step size used was 0.02.

Figure 3 shows the computed values of \( y \) at large values of \( \eta \) as a function of \( \tilde{a} \). It is seen that \( y \) is proportional to \( 1/\tilde{a} \), which agrees with the ion-sheath analysis given in Sec. III. The results for \( y(\tilde{x}) \) and \( n(\tilde{x}) \), \( n_0(\tilde{x}) \) are given in Figs. 4 and 5.

Two limiting cases, \( \tilde{a} = 0 \) and \( \tilde{a} = \infty \), are also included for the purpose of indicating the general range of the solutions. The value of \( y(\tilde{x} = 1) = \frac{3}{2} \ln (1/\tilde{a}) \) for small \( \tilde{a} \) is found to approach 3.0, and this provides us with the value of \( C(\varepsilon = 0) \) discussed in Sec. III. The analysis for the limiting case \( \tilde{a} = \infty \) is straightforward and is given in Appendix I.

The spatial distribution of potential is given in Fig. 6. Here the variable \( \tilde{p} \) is defined by

\[
\tilde{p} = \frac{1}{a \tilde{x}} = (1 + \varepsilon) \frac{1}{a \tilde{x}} = (1 + \varepsilon) \frac{I}{I_R}.
\]

V. CURRENT–VOLTAGE CHARACTERISTICS

From (1.26) and (3.2), we have

\[
\tilde{x}_p = \frac{\varepsilon I}{1 + \varepsilon I_R}.
\]

\( \tilde{x}_p \) is the distance of the probe from the boundary of the sheath.
Therefore, we can consider Fig. 4 as a plot of non-dimensional ion current versus probe potential for various values of \( \bar{a} \). Since, from (4.5), we know that

\[
\bar{\mu} = (1 + \epsilon) \left( \frac{\tau_e}{\bar{h}_o} \right) \left( \frac{1}{\bar{x}_e} \right).
\]

The data on Fig. 4 are cross-plotted to yield curves of \( y_o(\bar{x}_e, \bar{p}_o) \) which are presented in Fig. 7. This is therefore the ion current–voltage characteristic for a given probe. Note that saturation of ion current at very high \( y_o \) occurs only for the limiting case of \( \bar{p}_o \rightarrow \infty \) (or \( \bar{a} \rightarrow 0 \)). For all finite \( \bar{p}_o \), the ion current increases as \( y_o \) increases.

Knowing the potential distribution \( y = y(\bar{x}, \bar{a}) \), we can compute the value of \( \mu \) from (2.6):

\[
\mu = \left[ (1 + \epsilon) \int_0^{\bar{x}_e} \exp(y) \, d\bar{x} \right]^{-1}.
\]

The values of \( \mu \) have been computed using the numerical solutions \( y(\bar{x}, \bar{a}) \). To be consistent with our initial assumption, the magnitude of \( \mu \) so computed must be small compared to unity. In Fig. 7, the contours of \( \mu = 0.01, 0.10, 0.15, \) and 0.25 are shown. It is seen that the assumption of small \( \mu \) for a fixed \( y_o \) is best when \( \bar{p}_o \) is not too large. For \( \bar{p}_o \rightarrow \infty \), \( \mu \) is proportional to \( y_o^{-1}(1 + \epsilon)^{-1} \), as can be verified from (5.3) by using (3.6) for \( y(x) \) in the integral.

The total current \( J \) collected by the probe is simply \( J = I - I_e \). Using (5.1) and (1.16), we obtain

\[
\frac{1}{1 + \epsilon} \frac{J}{I_R} = x_p(1 - \alpha \mu),
\]

where \( \alpha = \mu_e/\mu_i \) and is a known constant for a given plasma. The value of \( \mu \) at the floating potential (when \( J = 0 \)) is therefore simply

\[
\mu_{(at \ floating \ y_o)} = \frac{1}{\bar{x}} = \frac{\mu_i}{\mu_e}.
\]

With \( \mu \) (at floating \( y_o \)) known, the value of the floating potential for given values of \( \bar{p}_o \) can then be found from (5.3) if desired. If \( \alpha \) is a large number, then the value of the floating potential will be large also. In Fig. 7 curves for \( \mu \bar{x}_e \) as a function of \( y_o \), for various \( \bar{p}_o \) are also plotted (\( \epsilon = 0 \) is assumed). Thus for any given \( \alpha \) the total current versus probe potential characteristic can easily be constructed using (5.4).

In Fig. 8, we have plotted \( (\epsilon I/I_n)(r_p/h_o)^2 \) versus \( y_o \) for various \( \bar{p}_o \); these are shown as solid lines. The dotted lines are for \( (\epsilon I/I_n)(r_p/h_o)^2 \) versus \( y_o \) for various \( \bar{p}_o \) (\( \alpha = 1 \) is assumed for simplicity). Note that both \( (\epsilon I/I_n)(r_p/h_o)^2 \) and \( (\epsilon J/I_n)(r_p/h_o)^2 \) are independent of the undisturbed plasma density \( N_o \). Then in an experiment when \( T \) and \( T_e \) are known, Fig. 8 may be used to determine the value of \( \rho_p \) from which the value of \( N_o \) may be found.

For very large \( \bar{p}_o \), interpolations in Figs. 7 and 8 will be inaccurate since the curves are very flat.
However, the analytical results from Sec. III are most accurate in this region. Replacing $\bar{d}$ by $(\bar{x}, \bar{p})^{-1}$ in (3.23) and neglecting $\ln \bar{x}_p$ in comparison with $\ln \bar{p}_o$, we obtain

$$y_o = \frac{3}{4} \ln \bar{p}_o + C(\epsilon) + \frac{2}{3} \bar{p}_o \bar{x}_o \int_0^{\bar{x}_o} \left(1 - \frac{1}{x^3}\right) dx,$$

or

$$y_o = \frac{3}{4} \ln \bar{p}_o - C(\epsilon) \frac{2}{3} \bar{x}_o \int_0^{\bar{x}_o} \left(1 - \frac{1}{x^3}\right) dx = G(\bar{x}_o). \quad (5.6)$$

The integral on the right-hand side may be expressed in terms of elliptic integrals. However, it is simpler to compute it numerically; the result is plotted in Fig. 9. Thus the value of $y_o$ can be easily obtained if $y_o$, $\bar{p}_o$, and $\epsilon$ are known. It should be clear that (5.6) is valid only when the left-hand side is positive. Note that when $\bar{x}_o$ is close to unity $G(\bar{x}_o)$ is approximately $(2\sqrt{2/3})(\bar{x}_o - 1)$. For very large $\bar{x}_o$, $G(\bar{x}_o)$ is approximately $\frac{3}{4} \bar{x}_o^2$. From the latter we deduced that for a fixed $\bar{p}_o$ then as $y_o \to \infty$, we have

$$y_o = \frac{1}{(1 + \epsilon)^\frac{3}{4}} \bar{p}_o \bar{x}_o \sim \left(\frac{1}{\bar{d}}\right) \bar{p}_o,$$

or $a \sim O(1/\bar{p}_o \bar{y}_o)^\frac{1}{4}$ which was used in (2.1).

VI. PROBES AT ARBITRARY POTENTIALS

We shall in this section give a general analysis of the probes at an arbitrary potential. Eliminating $n$ and $n_o$ from (1.11) to (1.13) we obtain a fourth-order equation for $y$ after considerable manipulation:

$$\frac{dA}{dy} + A + \mu \left( \frac{dx}{dy} - \epsilon \frac{d^2x}{dy^2} \right) = 0$$

with

$$A = \epsilon a^4 x^4 y_{xx} + \frac{1}{y_x} - a^2 x^4 y_{xx}. \quad (6.1)$$

For later convenience we introduce the following transformation:

$$\bar{x} = \frac{1 + \mu}{1 + \epsilon} x,$$

$$\bar{a} = \left[ \frac{1 + \epsilon}{1 + \mu} \right]^\frac{1}{4} a,$$

$$\bar{A} = \frac{1 + \mu}{1 + \epsilon} A.$$

Then (6.1) becomes

$$\frac{d\bar{A}}{dy} + \bar{A} + \mu \left( \frac{dx}{dy} - \epsilon \frac{d^2x}{dy^2} \right) = 0$$

and

$$\bar{A} = \frac{\epsilon \bar{a}^4 (\bar{x}) y_{xx} + 1}{y_x} - a^2 \bar{x}^4 y_{xx}. \quad (6.3)$$

Quasi-Neutral

In the limit of small $\bar{a}$, we regain the quasi-neutral solutions as given in (2.3) and (2.4).

$$n = n_o = 1 - \bar{x} + O(\bar{a}^2),$$

$$y = \frac{1 - \mu \epsilon}{1 + \mu} \ln (1 - \bar{x}) + O(\bar{a}^2). \quad (6.4)$$

This solution, as before, breaks down near $\bar{x} = 1$.

Transitional Layer

The analysis of the transitional layer is exactly as that in Sec. III. We introduce the following stretching transformation for the independent variable $\bar{x}$:

$$\bar{x} = 1 + \bar{d} t. \quad (6.5)$$

Using this in (6.2) and dropping the terms of order $\bar{a}^4$, we found that the resulting equation can be integrated explicitly once with respect to $t$. The detailed analysis follows exactly as that in Sec. III [see Eq. (3.9)]. The constant of integration is determined just as before by matching this solution to the quasi-neutral solutions (6.4), and is found to be zero. The final equation which gives the correct behavior near $\bar{x} = 1$ is then

$$\frac{dB}{dy} + B = \mu (1 + \epsilon) \frac{1}{y_x},$$

with

$$B = ey_{xx} - \frac{1}{2} y_x^2 + (1 + \mu) t. \quad (6.6)$$

Up to this point the analysis is quite general. It includes the whole range of potential (positive or negative). To go further, one has to make a study of the following separate cases:
(1) Very negative (positive) probes potential \( \mu = 0 \). The probe surface does not terminate within the transitional layer. Analysis beyond the transitional layer must be considered. We have considered this in Sec. III.

(2) Moderate potential. The probe surface is located in the transitional layer. For a more detailed analysis and numerical result see reference 11.

(3) Nearly plasma potential. The analysis is exactly the same as in (2). Since the potential is small, we can linearize (6.6), i.e.,

\[
e_{y_{i+1}} + (1 + \mu)\epsilon y_i = \mu \epsilon - 1.
\]

Letting

\[
\tilde{\epsilon} = \left[ \frac{1 + \mu}{\epsilon} \right]^t t,
\]

\[
\tilde{y} = \frac{1 + \mu}{1 - \mu \epsilon} y, \quad \text{and} \quad \tilde{y}_i = w(\tilde{\epsilon}),
\]

we obtain

\[
w_{i+1} + \tilde{\epsilon} w + 1 = 0.
\]

The boundary conditions for (6.8) are

\[
w_i = 0 \text{ at } \tilde{\epsilon} = 0 \text{ on the probe},
\]

\[
w = 0 \text{ as } \tilde{\epsilon} \to -\infty \text{ to join on to the quasi-neutral solution}.
\]

Since the solution of (6.8) is independent of any parameter in the problem, we conclude that the variation of potential within the sheath is of order \((1 - \mu \epsilon)/(1 + \mu)\) [see the transformation in (6.7)]. Comparing this with the potential at the edge of the sheath which from (6.4) is of the order

\[
y_{\text{sheath edge}} = \frac{1 - \mu \epsilon}{1 + \mu} \ln \left[ \frac{(\epsilon \tilde{\epsilon})^{1/2}}{(1 + \epsilon)^{1/2}} \right],
\]

we have then

\[
y = \frac{1 - \mu \epsilon}{1 + \mu} \left\{ \ln \left[ \frac{(\epsilon \tilde{\epsilon})^{1/2}}{(1 + \epsilon)^{1/2}} \right] + O(1) \right\}.
\]

We see the change of the potential within the sheath is negligible in the limit of small \( \tilde{\epsilon} \). To obtain a current–voltage characteristic, we must relate the factor in front of the bracket in (6.9) to the current collected by the probe. For this purpose we first integrate (1.11) and (1.12), i.e.,

\[
n = \exp \left( \frac{y}{\epsilon} \right) \left[ 1 - \frac{1}{\epsilon} \int_0^x \exp \left( -\frac{y}{\epsilon} \right) dx \right],
\]

\[
n_e = \exp \left( -y \right) \left[ 1 - \mu \int_0^x \exp \left( y \right) dx \right].
\]

Under the assumption of small \( y \), we expand the exponentials and keep only the first-order terms, i.e.,

\[
n = \left( 1 + \frac{y}{\epsilon} \right) \left[ 1 - \frac{1}{\epsilon} \int_0^x y dx + O(y^2) \right],
\]

\[
n_e = \left( 1 - y \right) \left[ 1 - \mu x - \mu \int_0^x y dx + O(y^2) \right].
\]

Using the boundary condition on the probe, i.e.,

\[
n = n_e = 0 \text{ at } x = x_p,
\]

we obtain from (6.10)

\[
x_p = \frac{1}{\epsilon} \int_0^{x_p} y(x) dx \approx \epsilon \left[ 1 + \frac{1}{\epsilon} \int_0^x y(x) dx \right],
\]

\[
\mu \approx \left[ x_p + \int_0^{x_p} y(x) dx \right]^{-1}
\]

\[
\approx \frac{1}{\epsilon} \left[ 1 - \frac{1}{\epsilon} (1 + \epsilon) \int_0^x y(x) dx \right].
\]

From (6.11) and (6.12) we have

\[
x_p - 1 = \frac{1 - \epsilon \mu}{1 + \epsilon} \approx \frac{1}{\epsilon} \left( 1 - \epsilon \mu \right).
\]

Using the relation in (1.26) we have

\[
\frac{I}{I_R} - 1 = \frac{1}{\epsilon} \left( 1 - \epsilon \mu \right).
\]

By the definition of \( \mu \) the logarithmic term in (6.9) can be transformed into

\[
-\ln \left[ \frac{(\epsilon \tilde{\epsilon})^{1/2}}{(1 + \epsilon)^{1/2}} \right] = \frac{1}{\epsilon} \ln \left[ \left( 1 + \frac{1}{\epsilon} \right)^{\frac{1}{2}} \right].
\]

The current–voltage characteristic can then be derived from (6.9) and (6.13), i.e.,

\[
\frac{I}{I_R} = 1 + \frac{3y_p}{\epsilon \ln [(1 + 1/\epsilon)\rho_0]} + O \left( \frac{y_p}{\ln^2 \rho_0} \right).
\]

VII. SUMMARY AND DISCUSSION

The basic premises underlying the theoretical model being studied are the following:

(1) The phenomenon under study is collision-dominated so that a continuum description is meaningful.

(2) The plasma is only slightly ionized so that only collisions between neutral and charged particles are important. This implies that phenomenological coefficients such as \( D_1 \) and \( D_2 \) are independent of the local ionization levels. Furthermore, the plasma as a whole can be considered as quiescent since the neutral particles are not directly affected by the presence of the probe.

Based on the above premises, the physical problem

\[\text{footnote: I. Cohen, Phys. Fluids 6, 1492 (1963).}\]
is readily formulated, and what remains is a mathematical problem posed by (1.11), (1.12), (1.13), (1.17), and (1.18); i.e., the construction of the desired solutions (1.19). However, this proves to be a task of such complexity that in the present paper only two limiting cases are considered, namely, the highly negative probe potential case \( y_p = \frac{2}{3} \ln (1/\hat{a}) \gg 1 \), and the near plasma potential case \( y_p/\epsilon \ll 1 \). Both are studied under the general restriction that \( \rho_p > 1 \) (a or \( \hat{a} \) < 1).

For the highly negative probe potential case, advantage is taken of the fact that diffusion process for ions due to electrostatic body forces overwhelms that due to molecular diffusion. This condition prevails when \( y_p \gg 1 \). This has the consequence that Boltzmann’s electron density distribution (2.7) is a valid approximation in both the quasi-neutral and transitional regions. In obtaining (2.7) from (2.5) we require that

\[
\mu \int_0^2 \exp \left( \frac{y}{x} \right) dx = \int_0^2 \exp \left( \frac{y_p}{x} \right) dx \ll \frac{1}{1 + \epsilon}.
\]

Thus for \( \epsilon \leq 1.0 \) with \( y_p = \frac{2}{3} \ln (1/\hat{a}) \gg 1 \) this condition is easily satisfied. The magnitude of

\[
(1 + \epsilon) \mu \int_0^2 \exp \left( \frac{y}{x} \right) dx
\]

is the error we have committed in the transitional region and are prepared to accept in this limiting case.

From the analytical study presented in Sec. III (under the restriction that \( \rho_p \gg 1 \) or \( \hat{a} \ll 1 \)), two significant results emerge. The first significant result is the division of the regions of disturbances. It is found that the disturbances are divided into four distinct regions, namely, the quasi-neutral, the transitional, the ion-sheath and the ion-diffusion region. In the latter two regions the Boltzmann electron density distribution (2.7) is not valid, but it is shown that the effects of electrons in these regions are completely negligible. The precise boundary condition of ion density on the probe is found to be irrelevant to the problem. The second significant result which emerges is the understanding of the effects of the parameter, \( \epsilon \) which is associated with the highest-order term of the equation. By introducing the transformation (3.2), (3.3), it is found possible to pin down qualitatively the additional effects of \( \epsilon \). We have the following observations:

1. In the quasi-neutral region, the effects of \( \epsilon \) on \( y(\hat{x}) \) is \( O(\epsilon^2) \).

2. In the transitional region, the effects of \( \epsilon \) on \( y(\hat{x}) \) is \( O(\epsilon^3) \). Qualitatively, the value of \( y(\hat{x}; \epsilon) \) is higher than \( y(\hat{x}; \epsilon = 0) \) by an amount of order \( \epsilon \).

3. In the ion-sheath region, the value of \( y(\hat{x}, \epsilon) \) differs from that of \( y(\hat{x}, \epsilon = 0) \) by a constant of order \( \epsilon \), and the qualitative behavior of \( y(\hat{x}) \) is independent of \( \epsilon \) to \( O(\epsilon^3) \).

4. In the ion-diffusion region, \( \hat{a} \epsilon \) carries the responsibility for satisfying the ion-density boundary condition on the probe. It has practically no effect on \( y(x) \) or the value of \( \hat{x}_p \).

Based on these observations, the third-order term is dropped from (3.4) in our program of numerical solutions. This step is crucial to the success of our numerical integration scheme. The major reasons are the following:

It is desired to construct solutions \( y(\hat{x}, \hat{a}) \) which are independent of \( y_p \). This is not possible if the third-order term is retained. Also, in any step-by-step integration in the \( +\hat{x} \) direction, the third-order term will cause instability since it is responsible for the ion-diffusion layer. Furthermore, in most practical situations, the value of \( \epsilon \) is usually small.

For the other limiting case, \( y_p/\epsilon \ll 1 \), the problem can be linearized. Physically, the electric energy per charged particle is small compared with its thermal energy, and the process of molecular diffusion dominates. The range of validity of that class of solutions is rather limited, but it is interesting to note that the potential distribution in the plasma is still confined to a narrow region near the probe.

In the intermediate range of \( y_p \), i.e., \( \epsilon \leq y_p \leq \frac{2}{3} \ln (1/\hat{a}) \), no calculation has been performed.\(^{12}\) In this range of \( y_p \), the values of \( \mu \) will not be negligible, and the Boltzmann electron density distribution (2.7) can no longer be used. However, it can be expected that the ion-sheath region will disappear, the transitional region will join directly with the ion-diffusion region; the value of \( \hat{x}_p \) and \( \hat{x}_s \) can thus be expected to be very close to each other, with the latter always smaller than the former. It is expected that the probe will terminate within the transitional layer. A similar “boundary layer” analysis as in Sec. III can readily be carried out (see Sec. VI). It should be noted also that the same method can be applied to positive probe potential as well. The loss of the Boltzmann electron density distribution does not introduce any difficulty.

The boundary condition for electron density at the probe is always important for any \( y_p > 0 \). It should be pointed out that, unlike the ion density

\(^{12}\) This problem is treated in reference 11. See Sect. VI of the present paper.
distribution, the electron density distribution does not behave in a boundary-layer manner near the probe when \( y_e > 0 \). In other words, a change in the electron density boundary condition will affect the solution everywhere. This conclusion is evident from (2.5).

The value of \( \bar{x}_e \) for the highly negative potential case is given implicitly from the approximate formula (5.6)

\[
\frac{y_e - \frac{3}{2} \ln \bar{p}_e - C(\epsilon)}{\bar{p}_e} = G(\bar{x}_e).
\]

It is seen that \( \bar{x}_e \) depends only weakly on \( \epsilon \). However, from (5.1), the ion current is given by

\[
I = I_e (1 + 1/\epsilon) \bar{x}_e. \tag{7.2}
\]

We thus see that the ion current is strongly affected by \( \epsilon \) especially when \( \epsilon \) is small. Physically, this is because when \( \epsilon \) is small, the ions have relatively low thermal energy, and are easily attracted to the probe. For the near plasma potential case, we have

\[
\frac{I}{I_e} = 1 + \frac{y_e}{\frac{3}{2} \ln \left[ (1 + 1/\epsilon) \rho_e \right]}. \tag{7.2}
\]

It is seen that the same strong dependence on \( \epsilon \) occurs.

It is interesting at this point to discuss the validity of the continuum assumption underlying the present theory. Formally, it is expected that the continuum assumption is valid provided that the characteristic mean free path of the charged particles \( l \) is small compared with a characteristic length of the problem. The appropriate characteristic length here must be the distance from the probe surface to the point where the quasi-neutral solutions begins to take over. Denoting this point by subscript \( * \), the following results are easily deduced:

\[
\frac{r_e - r_p}{\rho_e} = \rho_e (\bar{x}_e - 1) \sim \frac{y_e - \frac{3}{2} \ln \bar{p}_e - C(\epsilon)}{\bar{p}_e} \tag{7.3}
\]

highly negative probe potential,

\[
\frac{r_e - r_p}{\rho_e} \approx O(\rho_e)^{1/2} \tag{7.4}
\]

moderate potential,

\[
\frac{r_e - r_p}{\rho_e} \approx [\rho_e (1 + \epsilon^{-1})]^{1/2} \tag{7.5}
\]

near plasma potential.

Thus, for highly negative probe potential, the region of disturbance for a given probe size is proportional to the square root of the probe potential. The continuum assumption, i.e., the mean free path should be small compared to the thickness of this region of disturbance, can hardly be a problem. Furthermore, for a given range of potential, \( \bar{x}_e - 1 = O(1) \) we see from (7.3) that the physical distance of the disturbance in case of high probe potential is of order of probe size \( r_p \). For the near-plasma potential case, which is the most severe case if \( \epsilon \ll 1 \), the continuum assumption is formally valid if

\[
\frac{r_e - r_p}{\rho_e} \approx \frac{r_e}{\rho_e} \gg \frac{l}{\rho_e} \tag{7.3}
\]

or

\[
[(1 + \epsilon)(l/r_p)(l/h_i)^2] \ll 1,
\]

where \( h_i \) is Debye distance based on ion temperature, i.e., \( h_i^2 = kT/4\pi n_i e^2 Z \). Note that \( l/h_i \) is a plasma property. It is interesting to note that even when \( l/h_i \) is of order 1.0 or larger, the continuum theory can still be applied provided that the probe is large enough. The thickness of the ion-diffusion layer can be shown to be \( O(1 + \epsilon)^{1/2} \rho_e \). But since it was shown that for the highly negative potential case this layer has little effect, a breakdown of the continuum assumption here is therefore immaterial.

From (7.3) to (7.5), it is seen that the thickness of the sheath, which may be defined as the main potential drop region, is not of order of Debye length whenever \( \rho_e \gg 1 \).

In conclusion, it should be pointed out that at extremely high negative potentials, the drift velocity of the ions may become comparable to its thermal velocities. Ionization of neutral particles from collisions will then become important, and the basic assumption that the plasma is quiescent will break down. The results of the present theory must therefore be interpreted with this in mind.

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APPENDIX I. SMALL PROBE SOLUTION (\( \rho_e < 1 \))

The governing equations are

\[
a^2 x \frac{d^2 y}{dx^2} = n - n_*, \tag{A1}
\]

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13 For moderate probe potentials, the sheath thickness can be shown to be of order of Debye length based on the value of charge density at the edge of the sheath.
\[
\frac{dn_x}{dx} + \frac{dy}{dx} n_x = -\mu, \quad (A2)
\]
\[
\frac{dn_x}{dx} - \frac{dy}{dx} n_x = -(\epsilon - 1). \quad (A3)
\]

For \(\rho_p \ll 1\), we have from (1.25) \(a \gg 1\). Thus the solution of (A1), neglecting the right-hand side, is

\[
dy/dx = \text{const} = A. \quad (A4)
\]

Integrating this once and noting \(y = 0\) at \(x = 0\), we have

\[
y = Ax. \quad (A5)
\]

Now the boundary conditions for (A2) and (A3) are then

\[
y = y_0, \quad n = n_0 = 0, \quad (A6)
\]
\[
y = 0, \quad n = n_0 = 1. \quad (A7)
\]

These four conditions are just enough for the solution of (A2) and (A3) and the constant parameters, \(\mu\), and \(A\). Solving (A2) and (A3) with the fact of (A4), we have

\[
n_x = \{1 - (\mu/A) \{\exp (y) - 1\}\} \exp (-y), \quad (A8)
\]

which reduces to (A5) when \(x \gg 1/a\).