Lecture 01: One Period Model

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Overview

1. Securities Structure
   - Arrow-Debreu securities structure
   - Redundant securities
   - Market completeness
   - Completing markets with options

2. Pricing (no arbitrage, state prices, SDF, EMM …)

3. Optimization and Representative Agent
   (Pareto efficiency, Welfare Theorems, …)
The Economy

• State space (Evolution of states)
  - Two dates: $t=0, 1$
  - $S$ states of the world at time $t=1$

• Preferences
  - $U(c_0, c_1, \ldots, c_S)$
  - $MRP_{s,0}^A = -\frac{\partial U^A}{\partial c_s^A} / \frac{\partial U^A}{\partial c_0^A}$ (slope of indifference curve)

• Security structure
  - Arrow-Debreu economy
  - General security structure
Security Structure

- Security $j$ is represented by a payoff vector
  \[
  (x_1^j, x_2^j, \ldots, x_S^j)
  \]

- Security structure is represented by payoff matrix

\[
X = \begin{pmatrix}
  x_1^j & x_2^j & \cdots & x_{S-1}^j & x_S^j \\
  x_1^{j+1} & x_2^{j+1} & \cdots & x_{S-1}^{j+1} & x_S^{j+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_1^{J-1} & x_2^{J-1} & \cdots & x_{S-1}^{J-1} & x_S^{J-1} \\
  x_1^J & x_2^J & \cdots & x_{S-1}^J & x_S^J 
\end{pmatrix}
\]

- NB. Most other books use the transpose of $X$ as payoff matrix.
Arrow-Debreu Security Structure in $R^2$

One A-D asset $e_1 = (1, 0)$

This payoff cannot be replicated!

⇒ Markets are incomplete
Arrow-Debreu Security Structure in $R^2$

Add second A-D asset $e_2 = (0,1)$ to $e_1 = (1,0)$
Arrow-Debreu Security Structure in $R^2$

Add *second* A-D asset $e_2 = (0, 1)$ to $e_1 = (1, 0)$

Payoff space $<X>$

*Any payoff can be replicated with two A-D securities*
Arrow-Debreu Security Structure in $R^2$

Add **second** asset $(1,2)$ to \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

*Payoff space $\langle X \rangle$*

*New asset is **redundant** – it does not enlarge the payoff space*
Arrow-Debreu Security Structure

\[ X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \]

- \( S \) Arrow-Debreu securities
- each state \( s \) can be insured individually
- All payoffs are linearly independent
- Rank of \( X = S \)
- Markets are complete
General Security Structure

Only bond \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

Payoff space \(<X>\)
General Security Structure

Only bond $x^{bond} = (1,1)$

Payoff space $<X>$

Can’t be reached

(1, 1)
General Security Structure

Add security \((2,1)\) to bond \((1,1)\)
General Security Structure

Add security \((2,1)\) to bond \((1,1)\)

- Portfolio of
  - buy 3 bonds
  - sell short 1 risky asset
General Security Structure

Payoff space $<X>$

Two assets span the payoff space

Market are complete with security structure

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Payoff space coincides with payoff space of

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$
General Security Structure

- Portfolio: vector \( h \in R^J \) (quantity for each asset)
- Payoff of Portfolio \( h \) is \( \sum_j h^j x^j = h'X \)
- Asset span
  \[ <X> = \{z \in IR^S : z = h'X \text{ for some } h \in IR^J \} \]
  - \(<X>\) is a linear subspace of \( R^S \)
  - Complete markets \(<X> = R^S\)
  - Complete markets if and only if \( rank(X) = S \)
  - Incomplete markets \( rank(X) < S \)
  - Security \( j \) is redundant if \( x^j = h'X \) with \( h^j = 0 \)
Introducing derivatives

- Securities: property rights/contracts
- Payoffs of derivatives derive from payoff of underlying securities
- Examples: forwards, futures, call/put options

- Question:
  Are derivatives necessarily redundant assets?
Forward contracts

- Definition: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today
- Futures contracts are same as forwards in principle except for some institutional and pricing differences
- A forward contract specifies:
  - The features and quantity of the asset to be delivered
  - The delivery logistics, such as time, date, and place
  - The price the buyer will pay at the time of delivery
### Reading price quotes

#### Index futures

<table>
<thead>
<tr>
<th>INDEX</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Daily change</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJ Industrial Average (CBOT)-$10 times average</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar 9991 9992 9555 9683 - 224 11150 7900 27,474</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>June 9865 9890 9685 9668 - 226 10551 9080 589</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Est vol: 21,000; vol Fri 17:07; open int 28,254, +.822.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>idc pr: Hi 9905.46; Lo 9677.54; Close 9607.03; -220.17.</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

| S&P 500 Index (CMIE)-$250 times Index      |      |       |       |              |
| Mar 112350 112370 109100 109530 - 2810 134960 94100 474,811 |
| June 111950 111950 109350 109730 - 2830 170550 95000 17,224 |
| Dec 111580 111580 110020 110390 - 2830 150070 96130 304  |
| Est vol: 79,914; vol Fri 65,250; open int 520,626, -701. |
| idc pr: Hi 1122.20; Lo 1092.25; Close 1094.44; -27.76. |

| Mini S&P 500 (CMIE)-$50 times index       |      |       |       |              |
| Mar 112325 112400 109100 109625 - 2025 117850 99650 100,297 |
| Vol Fri 193,620; open int 100,333, -4,791. |

| S&P Midcap 400 (CMIE)-$500 times Index    |      |       |       |              |
| Mar 502.70 504.00 492.75 493.95 - 10.95 560.00 412.95 13,453 |
| Est vol: 1,140; vol Fri 1,101; open int 1,453, -207. |
| idc pr: Hi 504.26; Lo 492.74; Close 493.38, -10.88. |

| Nikkei 225 Stock Average (CMIE)-$5 times index |      |       |       |              |
| Mar 9990 9700 9555 9550 - 130 14020 9245 15,750 |
| Est vol: 667; vol Fri 2,100; open int 15,817, -17. |
| idc pr: Hi 9809.82; Lo 9823.98; Close 9831.93, -159.50. |

| Nasdaq 100 (CMIE)-$100 times Index         |      |       |       |              |
| Mar 133550 134500 147300 146700 - 4800 189400 112000 51,803 |
| Est vol: 18,215; vol Fri 17,500; open int 51,812, +763. |
| idc pr: Hi 1528.30; Lo 1471.52; Close 1479.17, -48.98. |

- **Settlement price** (last transaction of the day)
- **Low** of the day
- **High** of the day
- The **open price**
- **Expiration month**
- **Daily change**
- **Lifetime high**
- **Lifetime low**
- **Open interest**
Payoff diagram for forwards

- Long and short forward positions on the S&R 500 index:
Forward vs. outright purchase

- Forward + bond = Spot price at expiration - $1,020 + $1,020
  = Spot price at expiration
Additional considerations (ignored)

- Type of settlement
  - Cash settlement: less costly and more practical
  - Physical delivery: often avoided due to significant costs

- Credit risk of the counter party
  - Major issue for over-the-counter contracts
    - Credit check, collateral, bank letter of credit
  - Less severe for exchange-traded contracts
    - Exchange guarantees transactions, requires collateral
Call options

- A non-binding agreement (right but not an obligation) to buy an asset in the future, at a price set today
- Preserves the upside potential (😊), while at the same time eliminating the unpleasant (😢) downside (for the buyer)
- The seller of a call option is obligated to deliver if asked
Definition and Terminology

- A **call option** gives the owner the right but not the obligation to **buy** the underlying asset at a predetermined price during a predetermined time period.
- Strike (or exercise) price: The amount paid by the option buyer for the asset if he/she decides to exercise.
- Exercise: The act of paying the strike price to buy the asset.
- Expiration: The date by which the option must be exercised or become worthless.
- Exercise style: Specifies when the option can be exercised.
  - European-style: can be exercised only at expiration date.
  - American-style: can be exercised at any time before expiration.
  - Bermudan-style: can be exercised during specified periods.
## Reading price quotes

### S&P500 Index options

<table>
<thead>
<tr>
<th>Strike</th>
<th>Vol.</th>
<th>Last</th>
<th>Net CHG.</th>
<th>Open Int.</th>
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</thead>
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<tr>
<td>Feb 1080 c</td>
<td>100</td>
<td>26.50</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Feb 1080 p</td>
<td>358</td>
<td>13</td>
<td>+ 8.00</td>
<td>5</td>
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<tr>
<td>Mar 1080 c</td>
<td>10</td>
<td>44</td>
<td>...</td>
<td>...</td>
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<tr>
<td>Mar 1080 p</td>
<td>17</td>
<td>21.40</td>
<td>+ 6.00</td>
<td>412</td>
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<tr>
<td>Feb 1090 c</td>
<td>4</td>
<td>19</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Feb 1090 p</td>
<td>141</td>
<td>15.80</td>
<td>+ 9.00</td>
<td>279</td>
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<tr>
<td>Mar 1090 c</td>
<td>270</td>
<td>32</td>
<td>...</td>
<td>302</td>
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<tr>
<td>Mar 1090 p</td>
<td>343</td>
<td>28</td>
<td>...</td>
<td>302</td>
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<td>Feb 1100 c</td>
<td>1,041</td>
<td>15</td>
<td>-16.20</td>
<td>6,763</td>
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<tr>
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<td>3,246</td>
<td>20.10</td>
<td>+ 11.80</td>
<td>26,497</td>
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<td>4,439</td>
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<td>Mar 1100 p</td>
<td>8,235</td>
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<td>30,294</td>
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<td>37</td>
<td>-15.00</td>
<td>1,728</td>
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<td>838</td>
<td>18</td>
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<td>Mar 1120 p</td>
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<tr>
<td>Apr 1120 c</td>
<td>150</td>
<td>33.50</td>
<td>-6.50</td>
<td>10</td>
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</tbody>
</table>
Diagrams for purchased call

- Payoff at expiration

- Profit at expiration

Payoff ($)

Profit ($)

S&O Index Price ($)

Index price = $1020

Profit = $-95.68

Index price = $1000

Purchased call

Long forward
Put options

• A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.
• The seller of a put option is obligated to buy if asked.
• Payoff/profit of a purchased (i.e., long) put:
  - Payoff = \( \max[0, \text{strike price} - \text{spot price at expiration}] \)
  - Profit = Payoff – future value of option premium
• Payoff/profit of a written (i.e., short) put:
  - Payoff = - \( \max[0, \text{strike price} - \text{spot price at expiration}] \)
  - Profit = Payoff + future value of option premium
A few items to note

• A call option becomes more profitable when the underlying asset appreciates in value

• A put option becomes more profitable when the underlying asset depreciates in value

• Moneyness:
  - In-the-money option: positive payoff if exercised immediately
  - At-the-money option: zero payoff if exercised immediately
  - Out-of-the-money option: negative payoff if exercised immediately
Option and forward positions

A summary

- Long forward
- Short forward
- Long call
- Short call
- Long put
- Short put
Options to Complete the Market

Stock’s payoff: \( x^j = (1, 2, \ldots, S) \) (= state space)

Introduce call options with final payoff at \( T \):
\[
C_T = \max\{S_T - E, 0\} = [S_T - E]^+
\]

\( C_{E=1} = (0, 1, 2, \ldots, S - 2, S - 1) \)
\( C_{E=2} = (0, 0, 1, \ldots, S - 3, S - 2) \)
\[\ldots\]
\( C_{E=S-1} = (0, 0, 0, \ldots, 0, 1) \)
Options to Complete the Market

Together with the primitive asset we obtain

\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & S-1 & S \\
0 & 1 & 2 & \ldots & S-2 & S-1 \\
0 & 0 & 1 & \ldots & S-3 & S-2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 2 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}
\]

Homework: check whether this market is complete.
Cost of Portfolio and Returns

- Price vector $p \in \mathbb{R}^J$ of asset prices
- Cost of portfolio $h$,
  \[ p \cdot h := \sum_j p^j h^j \]
- If $p^j \neq 0$ the (gross) return vector of asset $j$ is the vector
  \[ R^j = \frac{x^j}{p^j} \]
Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)

2. Pricing
   • LOOP, No arbitrage and existence of state prices
   • Market completeness and uniqueness of state prices
   • Pricing kernel $q^*$
   • Three pricing formulas (state prices, SDF, EMM)
   • Recovering state prices from options

3. Optimization and Representative Agent
   (Pareto efficiency, Welfare Theorems, …)
Pricing

• State space (evolution of states)
• (Risk) preferences
• Aggregation over different agents
• Security structure – prices of traded securities

Problem:

• Difficult to observe risk preferences
• What can we say about existence of state prices without assuming specific utility functions for all agents in the economy
Vector Notation

• Notation: $y, x \in \mathbb{R}^n$
  - $y \geq x \iff y^i \geq x^i$ for each $i=1,\ldots,n$.
  - $y > x \iff y \geq x$ and $y \neq x$.
  - $y >> x \iff y^i > x^i$ for each $i=1,\ldots,n$.

• Inner product
  - $y \cdot x = \sum_i y_i x_i$

• Matrix multiplication
Three Forms of No-ARBITRAGE

1. **Law of one price (LOOP)**
   If $h'X = k'X$ then $p \cdot h = p \cdot k$.

2. **No strong arbitrage**
   There exists no portfolio $h$ which is a strong arbitrage, that is $h'X \geq 0$ and $p \cdot h < 0$.

3. **No arbitrage**
   There exists no strong arbitrage nor portfolio $k$ with $k'X > 0$ and $p \cdot k \leq 0$. 
Three Forms of No-ARBITRAGE

- Law of one price is equivalent to every portfolio with zero payoff has zero price.

- No arbitrage $\Rightarrow$ no strong arbitrage
No strong arbitrage $\Rightarrow$ law of one price
Pricing

- Define for each $z \in <X>$,

$$q(z) := \{ p \cdot h : z = h'X \}$$

- If LOOP holds $q(z)$ is a single-valued and linear functional. (i.e. if $h'$ and $h'$ lead to same $z$, then price has to be the same)

- Conversely, if $q$ is a linear functional defined in $<X>$ then the law of one price holds.
Pricing

• LOOP ⇒ \( q(h'X) = p \cdot h \)

• A linear functional \( Q \) in \( R^S \) is a valuation function if \( Q(z) = q(z) \) for each \( z \in <X> \).

• \( Q(z) = q \cdot z \) for some \( q \in R^S \), where \( q^s = Q(e_s) \), and \( e_s \) is the vector with \( e_s^s = 1 \) and \( e_s^i = 0 \) if \( i \neq s \)

• \( e_s \) is an Arrow-Debreu security

• \( q \) is a vector of state prices
State prices $q$

- $q$ is a vector of state prices if $p = Xq$, that is $p^j = x^j \cdot q$ for each $j = 1, \ldots, J$.

- If $Q(z) = q \cdot z$ is a valuation functional then $q$ is a vector of state prices.

- Suppose $q$ is a vector of state prices and LOOP holds. Then if $z = h'X$ LOOP implies that
  
  $$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x^j_s q_s) h^j =$$
  
  $$= \sum_s (\sum_j x^j_s h^j) q_s = q \cdot z$$

- $Q(z) = q \cdot z$ is a valuation functional $\iff$ $q$ is a vector of state prices and LOOP holds.
State prices q

\[ p(1,1) = q_1 + q_2 \]
\[ p(2,1) = 2q_1 + q_2 \]

Value of portfolio (1,2)

\[ 3p(1,1) - p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2 = q_1 + 2q_2 \]
The Fundamental Theorem of Finance

• **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with $q >> 0$.

• **Proposition 1’**. Let $X$ be an $J \otimes S$ matrix, and $p \in R^J$. There is no $h$ in $R^J$ satisfying $h \cdot p \leq 0$, $h' X \geq 0$ and at least one strict inequality if, and only if, there exists a vector $q \in R^S$ with $q >> 0$ and $p = X q$.

No arbitrage $\iff$ positive state prices
Multiple State Prices $q$ & Incomplete Markets

What state prices are consistent with $p(1,1)$?

$p(1,1) = q_1 + q_2$

One equation – two unknowns $q_1, q_2$

There are (infinitely) many.

E.g. if $p(1,1) = 0.9$

$q_1 = 0.45, q_2 = 0.45$

or $q_1 = 0.35, q_2 = 0.55$
complete markets
incomplete markets
Eco 525: Financial Economics I

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One Period Model

Slide 1-45

\[ p = Xq^o \]

incomplete markets
Multiple \( q \) in incomplete markets

Many possible state price vectors s.t. \( p = X'q \).
One is special: \( q^* \) - it can be replicated as a portfolio.
Uniqueness and Completeness

• **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.

• If markets are not complete, then there exists $v \in R^S$ with $0 = Xv$. Suppose there is no arbitrage and let $q >> 0$ be a vector of state prices. Then $q + \alpha v >> 0$ provided $\alpha$ is small enough, and $p = X (q + \alpha v)$. Hence, there are an infinite number of strictly positive state prices.
The Three Asset Pricing Formulas

- **State prices**
  \[ p_j = \sum_s q_s x_{s,j} \]

- **Stochastic discount factor**
  \[ p_j = E[m x^j] \]

- **Martingale measure**
  \[ p_j = \frac{1}{1+r_f} E_{\tilde{\pi}} [x^j] \]

(reflect risk aversion by over(under)weighing the “bad(good)” states!)
Stochastic Discount Factor

\[ p^j = \sum_s q_s x_s^j = \sum_s \pi_s \frac{q_s}{\pi_s} x_s^j \]

• That is, stochastic discount factor \( m_s = \frac{q_s}{\pi_s} \) for all \( s \).

\[ p^j = E[mx^j] \]
Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_s}$
Equivalent Martingale Measure

- Price of any asset: \( p^j = \sum_s q_s x_s^j \)
- Price of a bond: \( p_{\text{bond}} = \sum_s q_s = \frac{1}{1 + r_f} \)

\[
\begin{align*}
p^j &= \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j \\
p^j &= \frac{1}{1 + r_f} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j \\
&=: \hat{\pi}_s \\
p^j &= \frac{1}{1 + r_f} E_{\hat{\pi}} [x^j]
\end{align*}
\]
The Three Asset Pricing Formulas

- **State prices**
  \[ p^j = \sum_s q^j_s x^j_s \]

- **Stochastic discount factor**
  \[ p^j = E[m x^j] \]

- **Martingale measure**
  \[ p^j = 1/(1+r^f) E_{\tilde{\pi}} [x^j] \]
  (reflect risk aversion by over(under)weighing the “bad(good)” states!)
specify Preferences & Technology

observe/specify existing Asset Prices

State Prices $q$
(or stochastic discount factor/Martingale measure)

derive Asset Prices

• evolution of states
• risk preferences
• aggregation

absolute asset pricing

relative asset pricing

derive Price for (new) asset

Only works as long as market completeness doesn’t change
Recovering State Prices from Option Prices

- Suppose that $S_T$, the price of the underlying portfolio (we may think of it as a proxy for price of “market portfolio”), assumes a "continuum" of possible values.
- Suppose there are a “continuum” of call options with different strike/exercise prices $⇒$ markets are complete.
- Let us construct the following portfolio:
  for some small positive number $\varepsilon > 0$,
  - Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$
  - Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$
  - Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$
Recovering State Prices … (ctd.)

Value of the portfolio at expiration

Figure 8-2 Payoff Diagram: Portfolio of Options
Recovering State Prices ... (ctd.)

- Let us thus consider buying $\frac{1}{\epsilon}$ units of the portfolio. The total payment, when $\hat{S}_T - \frac{\delta}{2} \leq S_T \leq \hat{S}_T + \frac{\delta}{2}$, is $\epsilon \cdot \frac{1}{\epsilon} \equiv 1$, for any choice of $\epsilon$. We want to let $\epsilon \rightarrow 0$, so as to eliminate the payments in the ranges $S_T \in (\hat{S}_T - \frac{\delta}{2}, \hat{S}_T - \frac{\delta}{2})$ and $S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \epsilon)$. The value of $\frac{1}{\epsilon}$ units of this portfolio is:

$$\frac{1}{\epsilon} \left( C(S, E = \hat{S}_T - \frac{\delta}{2} - \epsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2}) - C(S, E = \hat{S}_T + \frac{\delta}{2} + \epsilon) \right] \right)$$
Taking the limit $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2}) \right] \right\}$$

$$= -\lim_{\varepsilon \to 0} \frac{C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2})}{-\varepsilon} + \lim_{\varepsilon \to 0} \frac{C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2})}{\varepsilon}$$

$$\leq 0$$

$$= -\frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2}) + \frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2})$$

Payoff

Divide by $\delta$ and let $\delta \to 0$ to obtain state price **density** as $\partial^2 C/\partial E^2$. 
Recovering State Prices … (ctd.)

Evaluating following cash flow

\[
\tilde{CF}_T = \begin{cases} 
0 & \text{if } S_T \notin \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right], \\
50000 & \text{if } S_T \in \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right].
\end{cases}
\]

The value today of this cash flow is:

\[
50000\left[\frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2}) - \frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2})\right]
\]

\[
q(S^1_T, S^2_T) = \frac{\partial C}{\partial E}(S, E = S^2_T) - \frac{\partial C}{\partial E}(S, E = S^1_T)
\]
### Table 8.1 Pricing an Arrow-Debreu State Claim

<table>
<thead>
<tr>
<th>E</th>
<th>C(S,E)</th>
<th>Cost of position</th>
<th>Payoff if $S_T =$</th>
<th>ΔC</th>
<th>$\Delta(\Delta C) = q_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3.354</td>
<td></td>
<td>7 8 9 10 11 12 13</td>
<td>-0.895</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.459</td>
<td></td>
<td></td>
<td>0.106</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.670</td>
<td>$+1.670$</td>
<td>0 0 0 1 2 3 4</td>
<td>-0.789</td>
<td>0.164</td>
</tr>
<tr>
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<td>1.045</td>
<td>$-2.090$</td>
<td>0 0 0 0 -2 -4 -6</td>
<td>-0.625</td>
<td>0.184</td>
</tr>
<tr>
<td>11</td>
<td>0.604</td>
<td>$+0.604$</td>
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<td>-0.441</td>
<td>0.162</td>
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<tr>
<td>12</td>
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<td>-0.279</td>
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<tr>
<td>13</td>
<td>0.164</td>
<td></td>
<td></td>
<td>-0.161</td>
<td>0.184</td>
</tr>
</tbody>
</table>

Total: 0.184
specify Preferences & Technology

observe/specify existing Asset Prices

• evolution of states
• risk preferences
• aggregation

relative asset pricing

absolute asset pricing

NAC/LOOP

derive Asset Prices

NAC/LOOP

State Prices q
(or stochastic discount factor/Martingale measure)

LOOP

derive Price for (new) asset

Only works as long as market completeness doesn’t change
Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)

2. Pricing
   (no arbitrage, state prices, SDF, EMM …)

3. Optimization and Representative Agent
   • Marginal Rate of Substitution (MRS)
   • Pareto Efficiency
   • Welfare Theorems
   • Representative Agent Economy
Representation of Preferences

A preference ordering is (i) complete, (ii) transitive, (iii) continuous [and (iv) relatively stable] can be represented by a utility function, i.e.

\[(c_0, c_1, \ldots, c_S) \succeq (c'_0, c'_1, \ldots, c'_S) \iff U(c_0, c_1, \ldots, c_S) > U(c'_0, c'_1, \ldots, c'_S)\]

(more on risk preferences in next lecture)
Agent’s Optimization

- Consumption vector \((c_0, c_1) \in \mathbb{R}^+_* \times \mathbb{R}^+_*\)
- Agent \(i\) has \(U^i : \mathbb{R}^+_* \times \mathbb{R}^+_* \to \mathbb{R}\)
- Endowments \((e_0, e_1) \in \mathbb{R}^+_* \times \mathbb{R}^+_*\)
- \(U^i\) is quasiconcave \(\{c : U^i(c) \geq v\}\) is convex for each real \(v\)
  - \(U^i\) is concave: for each \(0 \geq \alpha \geq 1\),
    \[U^i (\alpha c + (1- \alpha)c') \geq \alpha U^i (c) + (1-\alpha) U^i (c')\]
- \(\partial U^i/ \partial c_0 > 0, \partial U^i/ \partial c_1 >> 0\)
Agent’s Optimization

- Portfolio consumption problem

\[ \max_{c_0, c_1, h} U^i(c_0, c_1) \]
subject to (i) \[ 0 \leq c_0 \leq e_0 - p \cdot h \]
and (ii) \[ 0 \leq c_1 \leq e_1 + X' h \]

\[ U^i(c_0, \overrightarrow{c}_1) - \lambda [c_0 - e_0 + ph] - \overrightarrow{\mu} [\overrightarrow{c}_1 - \overrightarrow{e}_1 - h'X] \]

\[ \text{FOC} \]
\[ c_0 : \quad \frac{\partial U^i}{\partial c_0} (c^*) = \lambda \]
\[ c_s : \quad \frac{\partial U^i}{\partial c_s} (c^*) = \mu_s \]
\[ h : \quad \lambda \overrightarrow{p} = X \overrightarrow{\mu} \]
\[ \Leftrightarrow \quad \overrightarrow{p}^j = \sum_s \frac{\mu_s}{\lambda} x_t^j \]
Agent’s Optimization

\[ p^j = \sum_s \frac{\partial U^i / \partial c_s}{\partial U^i / \partial c_0} x^j_s \]

For time separable utility function

\[ U^i(c_0, \bar{c}) = u(c_0) + \delta u(\bar{c}) \]

and expected utility function (later more)

\[ U^i(c_0, \bar{c}_1) = u(c_0) + \delta E[u(c)] \]

\[ p^j = \sum_s \pi_s \delta \frac{\partial u^i / \partial c_s}{\partial u^i / \partial c_0} x^j_s \]
Welfare Theorems

• **First Welfare Theorem.** If markets are complete, then the equilibrium allocation is Pareto optimal.
  - State price is unique $q$. All $\text{MRS}_i(c^*)$ coincide with unique state price $q$.

• **Second Welfare Theorem.** Any Pareto efficient allocation can be decentralized as a competitive equilibrium.
Representative Agent & Complete Markets

- Aggregation Theorem 1: Suppose markets are complete

Then asset prices in economy with many agents are identical to an economy with a single agent/planner whose utility is

$$U(c) = \sum_k \alpha_k u^k(c),$$

where $\alpha^k$ is the welfare weights of agent $k$. and the single agent consumes the aggregate endowment.
Representative Agent & HARA utility world

• Aggregation Theorem 2: Suppose
  - riskless annuity and endowments are tradable.
  - agents have common beliefs
  - agents have a common rate of time preference
  - agents have LRT (HARA) preferences with

\[ R_A(c) = \frac{1}{A_i + Bc} \implies \text{linear risk sharing rule} \]

Then asset prices in economy with many agents are identical to a single agent economy with HARA preferences with

\[ R_A(c) = \frac{1}{\sum_i A_i + B}. \]
Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)

2. Pricing (no arbitrage, state prices, SDF, EMM …)

3. Optimization and Representative Agent
   (Pareto efficiency, Welfare Theorems, …)
Extra Material

Follows!
Portfolio restrictions

• Suppose that there is a short-sale restriction
  \[ h \in C = \{ h : h^j \geq -b^j, j \in J_0 \} \]
  where \( b \geq 0, J_0 \subseteq \{1, \ldots, J\} \)

• \( C \) is a convex set

• \( \langle \tilde{X} \rangle = \{ z \in R^S : z = h'X' \text{ for some } h \in C \} \)

• \( \langle \tilde{X} \rangle \) is a convex set

• For \( z \in \langle \tilde{X} \rangle \) let (cheapest portfolio replicating \( z \))
  \[ \tilde{q}(z) = \inf_h \{ p \cdot h : z = h'X', h \in C \} \]
Restricted/Limited Arbitrage

• An arbitrage is limited if it involves a short position in a security
  \[ j \in \mathcal{J}_0 \]

• In the presence of short-sale restrictions, security prices exclude (unlimited) arbitrage (payoff \( \infty \)) if, and only if, there exists a \( q \gg 0 \) such that
  \[ p^j \geq x^j \cdot q \quad \forall j \in \mathcal{J}_0 \]
  \[ p^j = x^j \cdot q \quad \forall j \notin \mathcal{J}_0 \]

• Intuition: \( q = MRS^i \) from optimization problem
  some agents wished they could short-sell asset
Portfolio restrictions (ctd.)

• As before, we may define $R^f = 1 / \sum_s q_s$, and $\tilde{\pi}_s$ can be interpreted as risk-neutral probabilities.
• $R^f p^j \geq E^\pi [x^j]$, with $= \text{if } j \notin J_0$
• $1/R^f$ is the price of a risk-free security that is not subject to short-sale constraint.
Portfolio restrictions (ctd.)

- Portfolio consumption problem
  \[ \max_{c_0, c_1, h} U^i(c_0, c_1) \]
  subject to (i) \( 0 \leq c_0 \leq w_0 - p \cdot h \)
  and (ii) \( 0 \leq c_1 \leq w_1 + h'X' \) and \( h \in \mathcal{J}_0 \)

- Proposition 4: Suppose \( c^* \gg 0 \) solves problem s.t. \( h^j \geq -b^j \) for \( j \in \mathcal{J}_0 \). Then there exists positive real numbers \( \lambda, \mu_1, \mu_2, \ldots, \mu_S \), such that
  - \( \frac{\partial U^i}{\partial c_0}(c^*) = \lambda \)
  - \( \frac{\partial U^i}{\partial c_0}(c^*) = (\mu_1, \ldots, \mu_S) \)
  - \( p^j \geq \sum_s \frac{\mu_s}{\lambda} x_s^j = \sum_s MR S_{0,s}^i x_s^j \)
  - \( p^j = \sum_s \frac{\mu_s}{\lambda} x_s^j \), if \( \not\in \mathcal{J}_0 \) or \( h^j > -b^j \)

The converse is also true.
FOR LATER USE Stochastic Discount Factor

\[ p^j = \sum_s \pi_s \delta \frac{\partial u^i(c^*)/\partial c_s}{\partial u^i(c^*)/\partial c_0} x^i_s \]

\[ m_s \]

\[ q_s \]

• That is, stochastic discount factor \( m_s = q_s/\pi_s \) for all \( s \).

\[ p^j = \sum_s \pi_s m_s x^j_s \]

\[ p^j = E[m x^j] \]