Lecture 02: Risk Preferences and Savings/Portfolio Choice

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State-by-state Dominance

- State-by-state dominance $\Rightarrow$ incomplete ranking
- « riskier »

Table 2.1 Asset Payoffs ($)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cost at $t=0$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s = 1</td>
<td>-1000</td>
<td>1050</td>
</tr>
<tr>
<td>s = 2</td>
<td>-1000</td>
<td>500</td>
</tr>
<tr>
<td>investment 1</td>
<td>-1000</td>
<td>1050</td>
</tr>
<tr>
<td>investment 2</td>
<td>-1000</td>
<td>500</td>
</tr>
<tr>
<td>investment 3</td>
<td>-1000</td>
<td>1050</td>
</tr>
<tr>
<td><strong>Value at $t=1$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_1 = \pi_2 = \frac{1}{2}$</td>
<td>s = 1</td>
<td>s = 2</td>
</tr>
<tr>
<td>s = 1</td>
<td>1200</td>
<td>1600</td>
</tr>
<tr>
<td>s = 2</td>
<td>1600</td>
<td>1600</td>
</tr>
</tbody>
</table>

- investment 3 state by state dominates 1.
State-by-state Dominance (ctd.)

Table 2.2 State Contingent ROR (r)

<table>
<thead>
<tr>
<th>State Contingent ROR (r)</th>
<th>s = 1</th>
<th>s = 2</th>
<th>Er</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment 1</td>
<td>5%</td>
<td>20%</td>
<td>12.5%</td>
<td>7.5%</td>
</tr>
<tr>
<td>Investment 2</td>
<td>-50%</td>
<td>60%</td>
<td>5%</td>
<td>55%</td>
</tr>
<tr>
<td>Investment 3</td>
<td>5%</td>
<td>60%</td>
<td>32.5%</td>
<td>27.5%</td>
</tr>
</tbody>
</table>

- Investment 1 mean-variance dominates 2
- BUT investment 3 does not m-v dominate 1!
State-by-state Dominance (ctd.)

Table 2.3 State Contingent Rates of Return

<table>
<thead>
<tr>
<th></th>
<th>State Contingent Rates of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s = 1</td>
</tr>
<tr>
<td>investment 4</td>
<td>3%</td>
</tr>
<tr>
<td>investment 5</td>
<td>3%</td>
</tr>
</tbody>
</table>

\[ \pi_1 = \pi_2 = \frac{1}{2} \]

\[ E[r_4] = 4\%; \quad \sigma_4 = 1\% \]
\[ E[r_5] = 5.5\%; \quad \sigma_5 = 2.5\% \]

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio \((E[r]-r^f)/\sigma\) than investment 5 for \(r^f = 0\).
Stochastic Dominance

- Stochastic dominance can be defined independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. ("risk-preference-free")
- Less "demanding" than state-by-state dominance
Stochastic Dominance

- Still incomplete ordering
  - “More complete” than state-by-state ordering
  - State-by-state dominance $\Rightarrow$ stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. (“risk-preference-free”)

- Next Section:
  - Complete preference ordering and utility representations

*Homework:* Provide an example which can be ranked according to FSD, but not according to state dominance.
### Table 3-1 Sample Investment Alternatives

<table>
<thead>
<tr>
<th>States of nature</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoffs</td>
<td>10</td>
<td>100</td>
<td>2000</td>
</tr>
<tr>
<td>Proba $Z_1$</td>
<td>.4</td>
<td>.6</td>
<td>0</td>
</tr>
<tr>
<td>Proba $Z_2$</td>
<td>.4</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

$$EZ_1 = 64, \; \sigma_{Z_1} = 44$$

$$EZ_2 = 444, \; \sigma_{Z_2} = 779$$
First Order Stochastic Dominance

**Definition 3.1**: Let $F_A(x)$ and $F_B(x)$, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval $[a,b]$. We say that $F_A(x)$ *first order stochastically dominates* ($FSD$) $F_B(x)$ if and only if for all $x \in [a,b]$

$$F_A(x) \leq F_B(x)$$

*Homework*: Provide an example which can be ranked according to $FSD$, but not according to state dominance.
First Order Stochastic Dominance
Table 3-2  Two Independent Investments

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Prob.</th>
<th>Payoff</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.25</td>
<td>1</td>
<td>0.33</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>6</td>
<td>0.33</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
<td>8</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Figure 3-6  Second Order Stochastic Dominance Illustrated
Definition 3.2: Let $F_A(\bar{x}), F_B(\bar{x})$ be two cumulative probability distribution for random payoffs in $[a, b]$. We say that $F_A(\bar{x})$ second order stochastically dominates $F_B(\bar{x})$ if and only if for any $x$:

$$\int_{-\infty}^{x} \left[ F_B(t) - F_A(t) \right] \, dt \geq 0$$

(with strict inequality for some meaningful interval of values of $t$).
Mean Preserving Spread

\[ x_B = x_A + z \]  \hspace{1cm} (3.8)

where \( z \) is independent of \( x_A \) and has zero mean

for normal distributions

\[ \mu = \int x \, f_A(x) \, dx = \int x \, f_B(x) \, dx \]

\( \tilde{x}, \text{Payoff} \)

Figure 3-7  Mean Preserving Spread
Mean Preserving Spread & SSD

- **Theorem 3.4**: Let $F_A(\cdot)$ and $F_B(\cdot)$ be two distribution functions defined on the same state space with identical means. Then the follow statements are equivalent:

  - $F_A(\bar{x})$ SSD $F_B(\bar{x})$
  - $F_B(\bar{x})$ is a mean preserving spread of $F_A(\bar{x})$ in the sense of Equation (3.8) above.
Expected Utility & Stochastic Dominance

- **Theorem 3.2**: Let $F_A(\tilde{x})$, $F_B(\tilde{x})$, be two cumulative probability distribution for random payoffs $\tilde{x} \in [a, b]$. Then $F_A(\tilde{x}) \ FSD \ F_B(\tilde{x})$ if and only if for all non-decreasing utility functions $U(\bullet)$.

$$E_A U(\tilde{x}) \geq E_B U(\tilde{x})$$
Expected Utility & Stochastic Dominance

- **Theorem 3.3**: Let $F_A(\tilde{x})$, $F_B(\tilde{x})$, be two cumulative probability distribution for random payoffs $\tilde{x}$ defined on $[a, b]$. Then, $F_A(\tilde{x})$ SSD $F_B(\tilde{x})$ if and only if $E_A U(\tilde{x}) \geq E_B U(\tilde{x})$ for all non-decreasing and concave $U$. 
Arrow-Pratt measures of risk aversion and their interpretations

- **absolute risk aversion**
  \[
  \rho_A(Y) = - \frac{U''(Y)}{U'(Y)} \equiv R_A(Y)
  \]

- **relative risk aversion**
  \[
  \rho_R(Y) = - \frac{Y U''(Y)}{U'(Y)} \equiv R_R(Y)
  \]

- **risk tolerance**
  \[
  \text{risk tolerance} = \frac{1}{R_A}
  \]
Absolute risk aversion coefficient

\[ R_A = -\frac{U''(Y)}{U'(Y)} \]

\[ \pi(Y, h) = \frac{1}{2} + \frac{1}{4} h R_A(Y) + HOT \]
Relative risk aversion coefficient

\[ R_R = -\frac{U''(Y)}{U'(Y)} Y \]

\[ \pi(Y, \theta) = \frac{1}{2} + \frac{1}{4}\theta R_R(Y) + HOT \]

*Homework:* Derive this result.
CARA and CRRA-utility functions

- Constant Absolute RA utility function
  \[ U(Y) = -e^{-\rho Y} \]

- Constant Relative RA utility function
  \[ U(Y) = \begin{cases} 
  \frac{Y^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\
  \ln Y & \text{for } \gamma = 1 
  \end{cases} \]
Investor’s Level of Relative Risk Aversion

\[
\frac{(Y + CE)^{1-\gamma}}{1-\gamma} = \frac{1}{2} \frac{(Y + 50,000)^{1-\gamma}}{1-\gamma} + \frac{1}{2} \frac{(Y + 100,000)^{1-\gamma}}{1-\gamma}
\]

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(CE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>75,000       (risk neutrality)</td>
</tr>
<tr>
<td>1</td>
<td>70,711</td>
</tr>
<tr>
<td>2</td>
<td>66,246</td>
</tr>
<tr>
<td>5</td>
<td>58,566</td>
</tr>
<tr>
<td>10</td>
<td>53,991</td>
</tr>
<tr>
<td>20</td>
<td>51,858</td>
</tr>
<tr>
<td>30</td>
<td>51,209</td>
</tr>
</tbody>
</table>

Y = 0

Y = 100,000

\(\gamma = 5\)  \(CE = 66,530\)
Risk aversion and Portfolio Allocation

- No savings decision (consumption occurs only at \( t=1 \))
- Asset structure
  - One risk free bond with net return \( r_f \)
  - One risky asset with random net return \( r \) (\( a = \)quantity of risky assets)

\[
\max_a E[U(Y_0(1 + r_f) + a(r - r_f))]
\]

FOC:
\[
E[U'(Y_0(1 + r_f) + a(r - r_f))(r - r_f)] = 0
\]
• **Theorem 4.1:** Assume $U'(\cdot) > 0$, and $U''(\cdot) < 0$ and let $\hat{a}$ denote the solution to above problem. Then

\[
\hat{a} > 0 \quad \text{if and only if} \quad E\tilde{r} > r_f
\]
\[
\hat{a} = 0 \quad \text{if and only if} \quad E\tilde{r} = r_f
\]
\[
\hat{a} < 0 \quad \text{if and only if} \quad E\tilde{r} < r_f
\]

• Define $W(a) = E\{U(Y_0(1+r_f) + a(\tilde{r} - r_f))\}$. The FOC can then be written

\[
W'(a) = E[U'(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)] = 0.
\]

By risk aversion ($U'' < 0$), $W''(a) = E[U''(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)^2] < 0$, that is, $W'(a)$ is everywhere decreasing. It follows that $\hat{a}$ will be positive if and only if $W'(0) = U'(Y_0(1+r_f))E(\tilde{r} - r_f) > 0$ (since then $a$ will have to be increased from its value of 0 to achieve equality in the FOC). Since $U'$ is always strictly positive, this implies $\hat{a} > 0$ if and only if $E(\tilde{r} - r_f) > 0$.

The other assertion follows similarly. □
Portfolio as wealth changes

- Theorem 4.4 (Arrow, 1971): Let \( \hat{a} = \hat{a}(Y_0) \) be the solution to max-problem above; then:

\[
\begin{align*}
(i) \quad & \frac{\partial R_A}{\partial Y} < 0 \quad \text{(DARA)} \quad \text{implies} \quad \frac{\partial \hat{a}}{\partial Y_0} > 0 \\
(ii) \quad & \frac{\partial R_A}{\partial Y} = 0 \quad \text{(CARA)} \quad \text{implies} \quad \frac{\partial \hat{a}}{\partial Y_0} = 0 \\
(iii) \quad & \frac{\partial R_A}{\partial Y} > 0 \quad \text{(IARA)} \quad \text{implies} \quad \frac{\partial \hat{a}}{\partial Y_0} < 0
\end{align*}
\]
Portfolio as wealth changes

- **Theorem 4.5 (Arrow 1971):** If, for all wealth levels $Y$,

  (i) $\frac{\partial R_R}{\partial Y} = 0$ (CRRA) implies $\eta = 1$

  (ii) $\frac{\partial R_R}{\partial Y} < 0$ (DRRA) implies $\eta > 1$

  (iii) $\frac{\partial R_R}{\partial Y} > 0$ (IRRA) implies $\eta < 1$

  where $\tilde{\eta} = \frac{\text{da/a}}{\text{dY/Y}}$ (elasticity)
Log utility & Portfolio Allocation

\[ U(Y) = \ln Y. \]

\[ E \left\{ \frac{\bar{r} - r_f}{Y_0 (1 + r_f) + a(\bar{r} - r_f)} \right\} = 0 \]

2 states, where \( r_2 > r_f > r_1 \)

\[ \frac{a}{Y_0} = \frac{(1 + r_f)[E[\bar{r}] - r_f]}{-(r_1 - r_f)(r_2 - r_f)} > 0 \]

Constant fraction of wealth is invested in risky asset!
Portfolio of risky assets as wealth changes

Now -- many risky assets

- Theorem 4.6 (Cass and Stiglitz, 1970). Let the vector
  \[
  \begin{bmatrix}
  \hat{a}_1(Y_0) \\
  \vdots \\
  \hat{a}_J(Y_0)
  \end{bmatrix}
  \]
  denote the amount optimally invested in the \(J\) risky assets if
  the wealth level is \(Y_0\). Then
  \[
  \begin{bmatrix}
  \hat{a}_1(Y_0) \\
  \vdots \\
  \hat{a}_J(Y_0)
  \end{bmatrix} = \begin{bmatrix}
  a_1 \\
  \vdots \\
  a_J
  \end{bmatrix}
  \]
  if and only if either
  (i) \(U'(Y_0) = (\theta Y_0 + \kappa)^\Delta\)
  (ii) \(U'(Y_0) = \xi e^{-\nu Y_0}\).

- In words, it is sufficient to offer a mutual fund.
LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion
  \[- \frac{u''(c)}{u'(c)} = \frac{1}{A + Bc}\]

- Special Cases
  - $B=0, A>0$    CARA
  - $B \neq 0, \neq 1$ Generalized Power
    - $B=1$   Log utility
    - $B=-1$  Quadratic Utility
    - $B \neq 1$ $A=0$   CRRA Utility function
    \[u(c) = \frac{1}{B-1} (A + Bc)^{\frac{B-1}{B}}\]
Prudence and Pre-cautionary Savings

• Introduce savings decision
  Consumption at $t=0$ and $t=1$

• Asset structure
  – NO risk free bond
  – One risky asset with random gross return $R$
Prudence and Savings Behavior

- Risk aversion is about the willingness to insure …
- … but not about its comparative statics.
- How does the behavior of an agent change when we marginally increase his exposure to risk?
- An old hypothesis (going back at least to J.M. Keynes) is that people should save more now when they face greater uncertainty in the future.
- The idea is called precautionary saving and has intuitive appeal.
Prudence and Pre-cautionary Savings

- Does not directly follow from risk aversion alone.
- Involves the third derivative of the utility function.
- Kimball (1990) defines absolute prudence as

\[ P(w) := -\frac{u''(w)}{u'''(w)}. \]

- Precautionary saving if any only if they are prudent.
- This finding is important when one does comparative statics of interest rates.
- Prudence seems uncontroversial, because it is weaker than DARA.
Pre-cautionary Saving

$$\max_s E[U(Y_0 - s) + \delta U(sR)]$$

s.t. $s \geq 0$

FOC: $U'(Y_0 - s) = \delta E[U'(sR)R]$

Is saving $s$ increasing/decreasing in risk of $R$?
Is RHS increasing/decreasing is riskiness of $R$?
Is $U'()$ convex/concave?
Depends on third derivative of $U()$!

N.B: For $U(c) = \ln c$, $U'(sR)R = 1/s$ does not depend on $R$. 
Pre-cautionary Saving

2 effects: Tomorrow consumption is more volatile
- consume more today, since it’s not risky
- save more for precautionary reasons

Theorem 4.7 (Rothschild and Stiglitz, 1971): Let \( \tilde{R}_A \) and \( \tilde{R}_B \) be two return distributions with identical means such that \( \tilde{R}_B = \tilde{R}_A + e \), (where \( e \) is white noise) and let \( s_A \) and \( s_B \) be, respectively, the savings out of \( Y_0 \) corresponding to the return distributions \( \tilde{R}_A \) and \( \tilde{R}_B \)

- If \( \bar{R}'_R(Y) \leq 0 \) and \( R_R(Y) > 1 \), then \( s_A < s_B \);
- If \( \bar{R}'_R(Y) \geq 0 \) and \( R_R(Y) < 1 \), then \( s_A > s_B \)
Prudence & Pre-cautionary Saving

\[ P(c) = \frac{-U'''(c)}{U''(c)} \]

\[ P(c)c = \frac{-cU'''(c)}{U''(c)} \]

- **Theorem 4.8**: Let \( \tilde{R}_A, \tilde{R}_B \) be two return distributions such that \( \tilde{R}_A \) SSD \( \tilde{R}_B \), and let \( s_A \) and \( s_B \) be, respectively, the savings out of \( Y_0 \) corresponding to the return distributions \( \tilde{R}_A \) and \( \tilde{R}_B \). Then,

\[
\begin{align*}
  s_A \geq s_B & \quad \text{iff } cP(c) \leq 2, \text{ and conversely,} \\
  s_A < s_B & \quad \text{iff } cP(c) > 2
\end{align*}
\]
Joint saving-portfolio problem

- Consumption at $t=0$ and $t=1$. (savings decision)
- Asset structure
  - One risk free bond with net return $r_f$
  - One risky asset ($a =$ quantity of risky assets)

\[
\max_{\{a,s\}} U(Y_0 - s) + \delta EU(s(1 + r_f) + a(\tilde{r} - r_f)) \quad (4.7)
\]

**FOC:**
\[
\begin{align*}
  s: & \quad U'(c_t) = \delta E[U'(c_{t+1})(1+r_f)] \\
  a: & \quad E[U'(c_{t+1})(r-r_f)] = 0
\end{align*}
\]
for CRRA utility functions

\begin{align*}
s &: \quad (Y_0 - s)^{\gamma}(-1) + \delta E\left[ (s(1 + r_f) + a(\bar{r} - r_f))^{-\gamma}(1 + r_f) \right] = 0 \\
a &: \quad E\left[ (s(1 + r_f) + a(\bar{r} - r_f))^{-\gamma}(\bar{r} - r_f) \right] = 0
\end{align*}

Where $s$ is total saving and $a$ is amount invested in risky asset.
Multi-period Setting

- Canonical framework (exponential discounting)
  \[ U(c) = E[ \sum \beta^t u(c_t) ] \]
  - prefers earlier uncertainty resolution if it affects action
  - indifferent, if it does not affect action

- Time-inconsistent (hyperbolic discounting)
  Special case: \( \beta - \delta \) formulation
  \[ U(c) = E[u(c_0) + \beta \sum \delta^t u(c_t)] \]

- Preference for the timing of uncertainty resolution
  recursive utility formulation (Kreps-Porteus 1978)
Multi-period Portfolio Choice

\[
\max \{s_t, a_t\}_{t=0}^{T-1} \quad E\left[\sum_{t=0}^{T} \beta^t U(c_t)\right]
\]

s.t.
\[
c_T = s_{T-1}(1 + r_f) + a_{T-1}(r_T - r_f)
\]
\[
c_t + s_t \leq s_{t-1}(1 + r_f) + a_{t-1}(r_t - r_f)
\]
\[
c_0 + s_0 \leq Y_0
\]

**Theorem 4.10 (Merton, 1971):** Consider the above canonical multi-period consumption-saving-portfolio allocation problem. Suppose \( U() \) displays CRRA, \( r_f \) is constant and \( \{r\} \) is i.i.d. Then \( a/s_t \) is time invariant.
Digression: Preference for the timing of uncertainty resolution

\[
U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))])
\]

Early (late) resolution if \( W(P_1, \ldots) \) is convex (concave)

Marginal rate of temporal substitution \( \leq \) risk aversion