Asset Pricing under Asymmetric Information Rational Expectations Equilibrium

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A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models: (uninformed) market maker submits a supply schedule first
    - static
      ◦ uniform price setting
      ◦ limit order book analysis
  - dynamic sequential trade models with multiple trading rounds
- signalling models: informed traders move first, market maker second
Overview

- Competitive REE (Examples)
  - Preliminaries
    - LRT (HARA) utility functions in general
    - CARA Gaussian Setup
      - Certainty equivalence
      - Recall Projection Theorem/Updating
  - REE (Grossman 1976)
  - Noisy REE (Hellwig 1980)
- Allocative versus Informational Efficiency
- Endogenous Information Acquisition
Utility functions and Risk aversion

- utility functions $U(W)$.
- Risk tolerance, $1/\rho = \text{reciprocal of the Arrow-Pratt measure of absolute risk aversion}$

$$\rho(W) := -\frac{\partial^2 U/\partial W^2}{\partial U/\partial W}.$$  

- Risk tolerance is linear in $W$ if

$$\frac{1}{\rho} = \alpha + \beta W.$$  

- also called hyperbolic absolute risk aversion (HARA) utility functions.
### LRT(HARA)-Utility Functions

<table>
<thead>
<tr>
<th>Class</th>
<th>Parameters</th>
<th>$U(W) =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponential/CARA</td>
<td>$\beta = 0$, $\alpha = 1/\rho$</td>
<td>$- \exp{-\rho W}$</td>
</tr>
<tr>
<td>generalized power</td>
<td>$\beta \neq 1$</td>
<td>$\frac{1}{\beta-1} (\alpha + \beta W)^{(\beta-1)/\beta}$</td>
</tr>
<tr>
<td>a) quadratic</td>
<td>$\beta = -1$, $\alpha &gt; W$</td>
<td>$-(\alpha - W)^2$</td>
</tr>
<tr>
<td>b) log</td>
<td>$\beta = +1$</td>
<td>$\ln(\alpha + W)$</td>
</tr>
<tr>
<td>c) power/CRRA</td>
<td>$\alpha = 0$, $\beta \neq 1, -1$</td>
<td>$\frac{1}{\beta-1} (\beta W)^{(\beta-1)/\beta}$</td>
</tr>
</tbody>
</table>
Certainty Equivalent in CARA-Gaussian Setup

\[ U(W) = -\exp(-\rho W), \text{ hence } \rho = -\frac{\partial^2 U(W)/\partial(W)^2}{\partial U(W)/\partial W} \]

\[ E[U(W) \mid \cdot] = \int_{-\infty}^{+\infty} -\exp(-\rho W)f(W \mid \cdot) dW \]

where \( f(W \mid \cdot) = \frac{1}{\sqrt{2\pi\sigma^2_W}} \exp\left[-\frac{(W - \mu_W)^2}{2\sigma^2_W}\right] \)

Substituting it in

\[ E[U(W) \mid \cdot] = \frac{1}{\sqrt{2\pi\sigma^2_W}} \int_{-\infty}^{+\infty} -\exp\left(-\frac{\rho z}{2\sigma^2_W}\right) dW \]

where \( z = (W - \mu_W)^2 - 2\rho\sigma^2_W W \)
Certainty Equivalent in CARA-Gaussian Setup

Completing squares \( z = (W - \mu_W - \rho \sigma_W^2)^2 - 2\rho(\mu_W - \frac{1}{2}\rho \sigma_W^2)\sigma_W^2 \)

Hence, \( E[U(W) \mid \cdot] = -\exp[-\rho(\mu_W - \frac{1}{2}\rho \sigma_W^2)] \times \)

\[ \times \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_W^2} \exp\left( -\frac{(W - \mu_W - \rho \sigma_W^2)^2}{2\sigma_W^2} \right) dW \]

\[ = 1 \]

Trade-off is represented by

\( V(\mu_W, \sigma_W^2) = \mu - \frac{1}{2} \rho \sigma_W^2 \)
Certainty Equivalent in CARA-Gaussian Setup
More generally, multinomial random variables $\mathbf{w} \sim \mathcal{N}(0, \mathbf{\Sigma})$ with a positive definite (co)variance matrix $\mathbf{\Sigma}$. More specifically, $E[\exp(\mathbf{w}^T \mathbf{A}\mathbf{w} + \mathbf{b}^T \mathbf{w} + d)] = $

$$= |\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A}|^{-1/2} \exp\left[\frac{1}{2} \mathbf{b}^T (\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A})^{-1} \mathbf{\Sigma} \mathbf{b} + d\right],$$

where

- $\mathbf{A}$ is a symmetric $m \times m$ matrix,
- $\mathbf{b}$ is an $m$-vector, and
- $d$ is a scalar.

Note that the left-hand side is only well-defined if $(\mathbf{I} - 2\mathbf{\Sigma}\mathbf{A})$ is positive definite.
Demand for a Risky Asset

- 2 assets
<table>
<thead>
<tr>
<th>asset</th>
<th>payoff</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond (numeraire)</td>
<td>( R )</td>
<td>( e_0^i )</td>
</tr>
<tr>
<td>stock</td>
<td>( v \sim \mathcal{N}(E[v</td>
<td>\cdot], \text{Var}[v</td>
</tr>
</tbody>
</table>

- \( P_{x^i} + b^i = P_{z^i} + e_0^i \)

- final wealth is
  \[ W^i = b^i R + x^i v = (e_0^i + P(z^i - x^i)) R + x^i v \]
  - mean: \( (e_0^i + P(z^i - x^i)) R + x E[v|\cdot] \),
  - variance: \( (x^i)^2 \text{Var}[v|\cdot] \)
Demand for a Risky Asset

\[ V(\mu_W, \sigma^2_W) = \mu_W - \frac{1}{2} \rho^i \sigma^2_W \]  \hspace{2cm} (1)

\[ = (e^i_0 + Pz^i)R + x^i(E[v|\cdot] - PR) - \frac{1}{2} \rho^i \text{Var}[v|\cdot](x^i)^2 \]  \hspace{2cm} (2)

First order condition: \( E[v|\cdot] - PR - \rho \text{Var}[v|\cdot]x^i = 0 \)

\[ x^i(P) = \frac{E[v|\cdot] - PR}{\rho^i \text{Var}[v|\cdot]} \]

Remarks

- independent of initial endowment (CARA)
- linearly increasing in investor’s expected excess return
- decreasing in investors’ variance of the payoff \( \text{Var}[v|\cdot] \)
- decreasing in investors’ risk aversion \( \rho \)
- for \( \rho^i \rightarrow 0 \) investors are risk-neutral and \( x^i \rightarrow +\infty \) or \( -\infty \)
Symmetric Info - Benchmark Model setup:

- \( i \in \{1, \ldots, I\} \) (types of) traders
- CARA utility function with risk aversion coefficient \( \rho^i \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance.)
- all traders have the same information \( v \sim \mathcal{N}(\mu, \sigma^2_v) \)
- aggregate demand
  \[
  \sum_i \eta^i \tau_v \{ E[v] - PR \} = \frac{1}{I} \sum_i \frac{1}{\rho^i} \text{ (harmonic mean)}
  \]
- market clearing
  \[
  \eta \tau_v \{ E[v] - PR \} = X^{\text{supply}}
  \]

\[
P = \frac{1}{R} \left\{ E[v] - \frac{X^{\text{sup}}}{I \eta \tau_v} \right\}
\]

The expected excess payoff \( Q := E[v] - PR = \frac{1}{\eta \tau_v} \frac{X^{\text{sup}}}{I} \)
Symmetric Info - Benchmark

- Trader $i$’s equilibrium demand is

$$x^i(P) = \frac{\eta^i}{\eta} \frac{X^{sup}}{I}$$

- Remarks:
  - $\frac{\partial P}{\partial E[v]} = \frac{1}{R} > 0$
  - $\frac{\eta^i}{\eta}$ risk sharing of aggregate endowment

$$\frac{x^{i*}}{x^{i'*}} = \frac{\eta^i}{\eta^{i'}}$$

- no endowment effects
REE - Grossman (1976) Model setup:

- $i \in \{1, \ldots, I\}$ traders
- CARA utility function with risk aversion coefficient $\rho^i = \rho$
  (Let $\eta^i = \frac{1}{\rho^i}$ be trader $i$'s risk tolerance.)
- Information is dispersed among traders
  trader $i$'s signal is $S^i = v + \epsilon^i_S$, where $\epsilon^i_S \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_{\epsilon})$
REE - Grossman (1976)

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \alpha_S \bar{S}, \text{ where } \bar{S} = \frac{1}{l} \sum_{i} S^i \text{ (sufficient statistics)} \]

**Step 2: Derive posterior distribution**

\[ E[v|S^i, P] = E[v|\bar{S}] = \lambda E[v] + (1-\lambda)\bar{S} = \lambda E[v] + (1-\lambda)\frac{P - \alpha_0}{\alpha_S} \]

\[ \text{Var}[v|S^i, P] = \text{Var}[v|\bar{S}] = \lambda \text{Var}[v] \]

where \( \lambda := \frac{\text{Var}[\epsilon]}{l \text{Var}[v] + \text{Var}[\epsilon]} \)

**Step 3: Derive individual demand**

\[ x^{i,*}(P) = \frac{E[v|S^i, P] - P(1+r)}{\rho^i \text{Var}[v|S^i, P]} \]

**Step 4: Impose market clearing**

\[ \sum_{i} x^{i,*}(P) = X^{\text{supply}} \]
### Informational (Market) Efficiency

<table>
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<tr>
<th>Form</th>
<th>Empirical Literature</th>
<th>Theoretical Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>strong</strong></td>
<td>Price reflects all private and public information</td>
<td>Price aggregates/reveals all private signals</td>
</tr>
<tr>
<td><strong>semi strong</strong></td>
<td>all public information</td>
<td>sufficient statistic of signals</td>
</tr>
<tr>
<td><strong>weak</strong></td>
<td>only (past) price information</td>
<td>a noisy signal of pooled private info</td>
</tr>
<tr>
<td><strong>fully revealing</strong></td>
<td></td>
<td><strong>with</strong> one signal reveals suff. stat.</td>
</tr>
<tr>
<td><strong>informational efficient</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>partially revealing</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>privately revealing</strong></td>
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</tbody>
</table>
Informational (Market) Efficiency

- $\bar{S}$ sufficient statistic for all individual info sets $\{S^1, \ldots, S^I\}$.

- Illustration: If one can view price function as

$$P(\cdot) : \{S^1, \ldots, S^I\} \xrightarrow{g(\cdot)} \bar{S} \xrightarrow{f(\cdot)} P$$

- If $f(\bar{S})$ is invertible, then price is informationally efficient.
- If $f(\cdot)$ and $g(\cdot)$ are invertible, then price is fully revealing.
Remarks & Paradoxa

- Grossman (1976) solved it via “full communication equilibria” (Radner 1979’s terminology)
- ‘unique’ info efficient equilibrium (DeMarzo & Skiadas 1998)
- As \( I \to \infty \) (risk-bearing capacity), \( P \to \frac{1}{R} E[v] \)
- Grossman Paradox:
  Individual demand does not depend on individual signal \( S^i \)’s. How can all information be reflected in the price?
- Grossman-Stiglitz Paradox:
  Nobody has an incentive to collect information?
- individual demand is independent of wealth (CARA)
- in equilibrium individual demand is independent of price
- equilibrium is not implementable
Noisy REE - Hellwig 1980

Model setup:

- \( i \in \{1, \ldots, I\} \) traders
- CARA utility function with risk aversion coefficient \( \rho^i = \rho \)
  (Let \( \eta^i = \frac{1}{\rho^i} \) be trader \( i \)'s risk tolerance.)
- information is dispersed among traders
  trader \( i \)'s signal is \( S^i = v + \epsilon^i_S \), where \( \epsilon^i_S \sim^{ind} \mathcal{N}(0, (\sigma^i_{\epsilon})^2) \)
- noisy asset supply \( X^{\text{Supply}} = u \)
- Let \( \Delta S^i = S^i - E[S^i] \), \( \Delta u = u - E[u] \) etc.
Noisy REE - Hellwig (1980)

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \sum_{i} \alpha_i S^i \Delta S^i + \alpha_u \Delta u \]

**Step 2: Derive posterior distribution** let’s do it only half way through

\[ E[v|S^i, P] = E[v] + \beta^i_S(\alpha) \Delta S^i + \beta^i_P(\alpha) \Delta P \]

\[ \text{Var}[v|S^i, P] = \frac{1}{\tau^i_{[v|S^i,P]}} \quad (\text{independent of signal realization}) \]

**Step 3: Derive individual demand**

\[ x^i,*(P) = \eta^i \tau^i_{[v|S^i,P]} \{ E[v|S^i, P] - P(1 + r) \} \]
Noisy REE - Hellwig (1980)

**Step 4: Impose market clearing**

Total demand = total supply (let \( r = 0 \))

\[
\sum_i \eta^i \tau^i_{[v|S^i,P]}(\alpha) \{ E[v] + \beta_S^i(\alpha) \Delta S^i - \alpha_0 \beta_P^i(\alpha) + [\beta_P^i(\alpha) - 1]P \} = u
\]

... \( P(S^1, \ldots, S^I, u) = \)

\[
\sum_i \left( \eta^i \tau^i_{[v|S^i,P]}(\alpha) \right) \left[ E[v] - \alpha_0 \beta_P^i(\alpha) + \beta_S^i(\alpha) \Delta S^i \right] - E[u] - \Delta u
\]

\[
\frac{\sum_i (1 - \beta_P^i(\alpha)) \eta^i \tau^i_{[v|S^i,P]}(\alpha)}{\sum_i (1 - \beta_P^i(\alpha)) \eta^i \tau^i_{[v|S^i,P]}(\alpha)}
\]
Noisy REE - Hellwig (1980)

Step 5: Impose rationality

\[
\alpha_0 = \frac{\sum_i \left( \eta^i \tau^i_{v|S^i,P}(\alpha) \right) \left[ E[v] - \alpha_0 \beta^i_P(\alpha) \right] - E[u]}{\sum_i (1 - \beta^i_P(\alpha)) \eta^i \tau^i_{v|S^i,P}(\alpha)}
\]

\[
\alpha^i_S = \frac{\sum_i \left( \eta^i \tau^i_{v|S^i,P}(\alpha) \right) \beta^i_S(\alpha)}{\sum_i (1 - \beta^i_P(\alpha)) \eta^i \tau^i_{v|S^i,P}(\alpha)}
\]

\[
\alpha^i_u = \frac{-1}{\sum_i (1 - \beta^i_P(\alpha)) \eta^i \tau^i_{v|S^i,P}(\alpha)}
\]

Solve for root \( \alpha^* \) of the problem \( \alpha = G(\alpha) \).
Noisy REE - Hellwig 1980

Simplify model setup:

- All traders have identical risk aversion coefficient $\rho = 1/\eta$
- Error of all traders’ signals $\epsilon^i_S$ are i.i.d.

**Step 1: Conjecture price function** simplifies to

$$\Delta P = \alpha_S \sum_i^1 \frac{1}{i} \Delta S^i + \alpha_u \Delta u$$

**Step 2: Derive posterior distribution**

$$E[v|S^i, P] = E[v] + \beta_S(\alpha) \Delta S^i + \beta_P(\alpha) \Delta P$$

$$\text{Var}[v|S^i, P] = \frac{1}{\tau} \quad \text{(independent of signal realization)}$$

where $\beta'$’s are projection coefficients.
Noisy REE - Hellwig (1980)
previous fixed point system simplifies to

\[
\alpha_S = \frac{1}{\sum_i (1 - \beta_P(\alpha))} \beta_S(\alpha)
\]

\[
\alpha_u = \frac{-1}{\eta \tau(\alpha) \sum_i (1 - \beta_P(\alpha))}
\]

To determine \(\beta_S\) and \(\beta_P\), invert Co-variance matrix

\[
\Sigma (S^i, P) = \begin{pmatrix}
\sigma_V^2 + \sigma_\varepsilon^2 & \alpha_S \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) \\
\alpha_S \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) & \alpha_S^2 \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) + \alpha_u^2 \sigma_u^2
\end{pmatrix}
\]

\[
\Sigma^{-1} (S^i, P) = \frac{1}{D} \begin{pmatrix}
\alpha_S^2 \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) + \alpha_u^2 \sigma_u^2 & -\alpha_S \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) \\
-\alpha_S \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) & \sigma_V^2 + \sigma_\varepsilon^2
\end{pmatrix}
\]

\[
D = \alpha_S^2 \frac{l-1}{I} \left(\sigma_V^2 + \frac{1}{I} \sigma_\varepsilon^2\right) \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \left(\sigma_V^2 + \sigma_\varepsilon^2\right)
\]
Noisy REE - Hellwig (1980)
Since $\text{Cov}[v, P] = \alpha_S \sigma_v^2$ and $\text{Cov}[v, S^i] = \sigma_v^2$ leads us to

$$\beta_P = \frac{1}{D} \alpha_S \frac{l - 1}{l} \sigma_v^2 \sigma_{\epsilon}^2$$

$$\beta_S = \frac{1}{D} \alpha_u^2 \sigma_u^2 \sigma_v^2$$

For conditional variance (precision) from projection theorem.

$$\text{Var}[v|S^i, P] = \frac{1}{D} \left[ D \sigma_v^2 - \left( \alpha_u^2 \sigma_u^2 + \alpha_S^2 \frac{l - 1}{l} \sigma_{\epsilon}^2 \right) \sigma_v^4 \right]$$

$$= \frac{1}{D} \left[ \alpha_S^2 \frac{l - 1}{l^2} \sigma_{\epsilon}^2 + \alpha_u^2 \sigma_u^2 \right] \left( \sigma_{\epsilon}^2 \right) \sigma_v^2$$

Hence,
Noisy REE - Hellwig (1980)

\[ \alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{(D - \alpha_s \frac{l-1}{l} \sigma_e^2 \sigma_v^2) l} \]

\[ \alpha_u = -\rho \frac{\left( \alpha_u^2 \sigma_u^2 + \alpha_s^2 \frac{l-1}{l^2} \sigma_e^2 \right) \sigma_v^2 \sigma_e^2}{(D - \alpha_s \frac{l-1}{l} \sigma_e^2 \sigma_v^2) l} \]

Trick:
Solve for \( h = -\frac{\alpha_u}{\alpha_S} \). (Recall price signal can be rewritten as
\[ \frac{P - \alpha_0}{\alpha_S} = \sum_i \frac{1}{l} S + \frac{\alpha_u}{\alpha_S} u. \) [noise signal ratio]

\[ h = \frac{\rho \left( h^2 \sigma_u^2 + \frac{l-1}{l^2} \sigma_e^2 \right) \sigma_v^2 \sigma_e^2}{h^2 \sigma_v^2 \sigma_u^2} \]

\[ h > \rho \sigma_e^2 \]

\( h \) increasing in \( h \)
\( \frac{l-1}{l^2} \sigma_e^2 \) decreasing in \( h \)

\( h \) unique

Noisy REE - Hellwig (1980)
Remember that \( h \) is increasing in \( \rho \).
Back to \( \alpha_S \)
\[
\alpha_S = \frac{\alpha_u^2 \sigma_v^2 \sigma_u^2}{D - \alpha_S \frac{l-1}{l} \sigma^2 \sigma_v^2} \quad \text{multiply by denominator}
\]
\[
\alpha_S D = \alpha_u^2 \sigma_v^2 \sigma_u^2 + \alpha_S^2 \frac{l-1}{l} \sigma^2 \sigma_v^2 \iff \alpha_S = \frac{1}{D} \left[ \alpha_u^2 \sigma_v^2 \sigma_u^2 + \alpha_S^2 \frac{l-1}{l} \sigma^2 \sigma_v^2 \right]
\]
Sub in \( D = \ldots \)
\[
\alpha_S = \frac{\frac{\alpha_u^2}{\alpha_S^2} \sigma_v^2 \sigma_u^2 + \frac{l-1}{l} \sigma^2 \sigma_v^2}{\frac{l-1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma^2 \right) \sigma^2 + \frac{\alpha_u^2}{\alpha_S^2} \sigma_u^2 \left( \sigma_v^2 + \sigma^2 \right)} \quad \Rightarrow \text{unique } \alpha_S.
\]
This proves existence and uniqueness of the NREE!
Characterization of NREE

Recall that \( \text{Var} \left[ v \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_s^2 \frac{l-1}{l^2} \sigma^2_\varepsilon + \alpha_u^2 \sigma^2_u \right] \sigma^2_\varepsilon \sigma^2_v \)

and \( \alpha_s = \frac{1}{D} \left[ \alpha_u^2 \sigma^2_u + \alpha_s^2 \frac{l-1}{l} \sigma^2_\varepsilon \right] \sigma^2_v \)

Hence, \( \alpha_s = \text{Var} \left[ v \mid S^i, P \right] \frac{\left[ \alpha_u^2 \sigma^2_u + \alpha_s^2 \frac{l-1}{l} \sigma^2_\varepsilon \right]}{\left[ \alpha_s^2 \frac{l-1}{l^2} \sigma^2_\varepsilon + \alpha_u^2 \sigma^2_u \right]} \) (notice \( l^2 \) square)

\[
\alpha_s = \text{Var} \left[ v \mid S^i, P \right] \frac{\frac{l^2}{l-1} h^2 \sigma^2_u + (l-1) \sigma^2_\varepsilon}{\sigma^2_\varepsilon + \frac{l^2}{l-1} h^2 \sigma^2_u}
\]

\[
\text{Var} \left[ v \mid S^i, P \right] \frac{1}{\sigma^2_\varepsilon} \left[ \frac{l^2}{l-1} h^2 \sigma^2_u + \sigma^2_\varepsilon + (l-1) \sigma^2_\varepsilon \right] \frac{\sigma^2_v}{\sigma^2_\varepsilon + \frac{l^2}{l-1} h^2 \sigma^2_u}
\]

\[
\alpha_s = \text{Var} \left[ v \mid S^i, P \right] \frac{1}{\sigma^2_\varepsilon} \left[ 1 + \frac{(l-1) \sigma^2_\varepsilon}{\sigma^2_\varepsilon + \frac{l^2}{l-1} h^2 \sigma^2_u} \right]
\]

\[
= \text{Var} \left[ v \mid S^i, P \right] \tau_\varepsilon \left[ 1 + (l - 1) \frac{\tau_u}{\tau_u + \frac{l^2}{l-1} h^2 \tau_\varepsilon} \right]
\]

\[
:= \theta
\]

\[
\alpha_s = \text{Var} \left[ v \mid S^i, P \right] \tau_\varepsilon \left[ 1 + \theta \right] \theta \text{ is decreasing in } \rho \text{ (} h \text{ is increasing)}
\]
Characterization of NREE

\[
\text{Var} \left[ v \mid S^i, P \right] = \frac{1}{D} \left[ \alpha_S^2 \frac{l-1}{l^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \right] \sigma_\varepsilon^2 \sigma_v^2 = \frac{\alpha_S^2 \frac{l-1}{l^2} \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2}{\alpha_S^2 \frac{l-1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) \sigma_\varepsilon^2 + \alpha_u^2 \sigma_u^2 \left( \sigma_v^2 + \sigma_\varepsilon^2 \right)} = \frac{\left[ \frac{l-1}{l^2} \sigma_\varepsilon^2 + h^2 \sigma_u^2 \right] \sigma_\varepsilon^2 \sigma_v^2}{h^2 \frac{l-1}{l} \left( \sigma_v^2 + \frac{1}{l} \sigma_\varepsilon^2 \right) \sigma_\varepsilon^2 + h^2 \left( \sigma_v^2 + \sigma_\varepsilon^2 \right)} = \ldots
\]

\text{“price precision”}

\[
\frac{1}{\text{Var} \left[ v \mid S^i, P \right]} = \tau_v + \tau_\varepsilon + (l - 1) \theta \tau_\varepsilon
\]

\textbf{Interpretation}

\[
\theta = (l - 1) \frac{\tau_u}{\tau_u + \frac{1}{l-1} h^2 \tau_\varepsilon}
\]

\text{measure of info efficiency}

\[
\sigma_u^2 \to \infty \ (\tau_u \to 0): \ \theta \to 0 \ \text{price is uninformative (Walrasian equ.)}
\]

\[
\sigma_u^2 \to 0 \ (\tau_u \to \infty): \ \theta \to 1 \ \text{price is informationally efficient}
\]
Remarks to Hellwig (1980)

• Since $\alpha_u^2 \neq 0$, $\beta_S \neq 0$, i.e. agents condition on their signal as risk aversion of trader increases the informativeness of price $\theta$ declines.
• Price informativeness increases in precision of signal $\tau_{\varepsilon}$ and declines in the amount of noise trading $\sigma_u^2$.
• Negative supply shock leads to a larger price increase compared to a Walrasian equilibrium, since traders wrongly partially attribute it to a good realization of $v$.
• Diamond and Verrecchia (1981) is similar except that endowment shocks of traders serve as asymmetric information.
Endogenous Info Acquisition

Model setup:

• $i \in \{1, \ldots, I\}$ traders
• CARA utility function with risk aversion coefficient $\rho$
  (Let $\eta = \frac{1}{\rho}$ be traders’ risk tolerance.)
• no information aggregation - two groups of traders
  • informed traders who have the same signal $S = \nu + \epsilon_S$
    with $\epsilon_S \sim \mathcal{N}(0, \sigma^2_{\epsilon_S})$
  • uninformed traders have no signal
• FOCUS on information acquisition
Noisy REE - Grossman-Stiglitz

**Step 1: Conjecture price function**

\[ P = \alpha_0 + \alpha_S \Delta S + \alpha_u \Delta u \]

**Step 2: Derive posterior distribution**

- for informed traders:
  
  \[ E[v|S, P] = E[v|S] = E[v] + \frac{\tau \varepsilon}{\tau_v + \tau \varepsilon} \Delta S \]
  
  \[ \tau[v|S] = \tau_v + \tau \varepsilon \]

- for uninformed traders:
  
  \[ E[v|P] = E[v] + \frac{\alpha_S \sigma_v^2}{\alpha_S^2 (\sigma_v^2 + \sigma_\varepsilon^2) + \alpha_u^2 \sigma_\varepsilon^2} \Delta P \]

  \[ \text{Var}[v|P] = \sigma_v^2 (1 - \frac{\alpha_S \sigma_v^2}{\alpha_S^2 (\sigma_v^2 + \sigma_\varepsilon^2) + \alpha_u^2 \sigma_\varepsilon^2}) \text{ OR} \]

  \[ \tau[v|P] = \tau_v + \frac{\tau_u}{\tau_u + h^2 \tau \varepsilon}, \text{ where } h = -\frac{\alpha_u}{\alpha_S} \]

  \[ :=\phi \in [0,1] \]

  After some algebra we get \[ E[v|P] = E[v] + \frac{1}{\alpha_S} \frac{\phi \tau \varepsilon}{\tau_v + \phi \tau \varepsilon} \Delta P \]
Noisy REE - Grossman-Stiglitz

**Step 3: Derive individual demand**

\[
x^I (P, S) = \eta^I [\tau_v + \tau_\varepsilon] \left[ E [v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S - P \right]
\]

\[
x^U (P) = \eta^U [\tau_v + \phi \tau_\varepsilon] \left[ E [v] + \frac{1}{\alpha S} \frac{\phi \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon} \Delta P - P \right]
\]

**Step 4: Impose market clearing**

Aggregate demand, for a mass of \( \lambda^I \) informed traders and \( (1 - \lambda^I) \) uninformed

\[
\lambda^I \eta^I [\tau_v + \tau_\varepsilon] \left[ E [v] + \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S - P \right] + \underbrace{\left(1 - \lambda^I\right) \eta^U [\tau_v + \phi \tau_\varepsilon] \left[ E [v] + \frac{1}{\alpha S} \frac{\phi \tau_\varepsilon}{\tau_v + \phi \tau_\varepsilon} \Delta P - P \right]}_{:= \nu^I} = u
\]
Noisy REE - Grossman-Stiglitz

\[
P(S, u) = \frac{(\nu^l + \nu^U)E[v] + \nu^l \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon} \Delta S - \frac{1}{\alpha_s} \frac{\phi_\tau_\varepsilon}{\tau_v - \phi_\tau_\varepsilon} \alpha_0 \nu^U - E[u] - \Delta u}{\nu^U \left(1 - \frac{1}{\alpha_s} \frac{\phi_\tau_\varepsilon}{\tau_v - \phi_\tau_\varepsilon}\right) + \nu^l}
\]

Hence, \( h = -\frac{\alpha_u}{\alpha_S} = \left[\nu^l \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon}\right]^{-1} = \frac{1}{\lambda^l \eta^l \tau_\varepsilon} \).

Hence, \( \phi = \frac{\tau_\upsilon \tau_\varepsilon}{\tau_\upsilon \tau_\varepsilon + \left(\lambda^l \eta^l\right)^2} \).

Remarks:

- As Var \([u]\) \(\searrow\) 0, \( \phi \searrow 1 \)
- If signal is more precise (\( \tau_\varepsilon \) is increasing) then \( \phi \) increases (since informed traders are more aggressive)
- Increases in \( \lambda^l \) and \( \eta^l \) also increase \( \phi \)
Noisy REE - Grossman-Stiglitz

Step 5: Impose rationality
Solve for coefficients

\[ \alpha_0 = E[v] - \frac{1}{\nu I + \nu U} E[u] \]

\[ \alpha_S = \frac{1}{\nu U \left( 1 - \frac{1}{\alpha_S \tau - \phi \tau} \right) + \nu I \tau_v + \tau_{\epsilon}} \]

\[ \alpha_u = - \frac{1}{\nu I + \nu U} \left( 1 + \frac{\lambda U \tau U}{\lambda I \tau I} \phi \right) \]

Finally let’s calculate

\[ \frac{\tau[v|S]}{\tau[v|P]} = \frac{\tau_v + \tau_{\epsilon}}{\tau_v + \phi \tau_{\epsilon}} = 1 + \frac{(1 - \phi) \tau_{\epsilon}}{\tau_v + \phi \tau_{\epsilon}} \]
Information Acquisition Stage - Grossman-Stiglitz (1980)

- **Aim:** endogenize $\lambda^i$

- **Recall**
  \[ x^i = \eta^i \tau_{[Q|S]} E[Q|S], \text{ where } Q = \nu - RP \text{ is excess payoff} \]

- **Final wealth is**
  \[ W^i = \eta^i Q \tau_{[Q|S]} E[Q|S] + (P u^i + e_i^0) R \]
  (CARA $\Rightarrow$ we can ignore second term)

  Note $W^i$ is product of two normally distributed variables

  Use Formula of Slide 7 or follow following steps:
  Conditional on $S$, wealth is normally distributed.

  \[
  \begin{align*}
  E[W|S] &= \eta \tau_{[Q|S]} E[Q|S]^2 \\
  Var[W|S] &= \eta^2 \tau_{[Q|S]} E[Q|S]^2 
  \end{align*}
  \]

- **the expected utility conditional on $S$**
  \[
  E[U(W)|S] = - \exp \left\{ - \frac{1}{\eta} \left[ \eta \tau_{[Q|S]} E[Q|S]^2 - \frac{1}{2} \eta \tau_{[Q|S]} E[Q|S]^2 \right] \right\}
  \]
Information Acquisition Stage - Grossman-Stiglitz (1980)

\[ E[U(W)|S] = - \exp\left\{ -\frac{1}{2} \tau[Q|S] E[Q|S]^2 \right\} \]

Integrate over possible \( S \) to get the ex-ante utility.

W.l.o.g. we can assume that \( S = Q + \epsilon \).

Normal density \( \phi(S) = \sqrt{\frac{\tau S}{2\pi}} \exp\left\{ -\frac{1}{2} \tau S (\Delta S)^2 \right\} \)

\[ E[U(W)] = - \int_S \sqrt{\frac{\tau[S]}{2\pi}} \exp \left\{ -\frac{1}{2} \left[ \tau[Q|S] E[Q|S]^2 + \tau S (\Delta S)^2 \right] \right\} \, dS \]

Term in square bracket is
\[
\left[ (\tau_Q + \tau_\epsilon) \left( E[Q] + \frac{\tau_\epsilon}{\tau_Q + \tau_\epsilon} \Delta S \right)^2 + \frac{\tau_Q \tau_\epsilon}{\tau_Q + \tau_\epsilon} (\Delta S)^2 \right]
\]

simplifies to
\[
\tau_Q E[Q]^2 + \tau_\epsilon (\Delta S + E[Q])^2
\]
Information Acquisition Stage - Grossman-Stiglitz (1980)

Hence, \( E \left[ U (W) \right] = - \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \int_S \sqrt{\frac{\tau S}{2\pi}} \exp \left\{ - \frac{1}{2} \left[ \tau \epsilon (\Delta S + E [Q])^2 \right] \right\} ds \)

Define \( y := \sqrt{\tau \epsilon} (\Delta S + E [Q]) \)

\[
E \left[ U (W) \right] = - \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \sqrt{\frac{\tau S}{\tau \epsilon}} \int_S - \sqrt{\frac{\tau \epsilon}{2\pi}} \exp \left\{ - \frac{1}{2} y^2 \right\} ds
\]

Letting \( k = - \exp \left\{ - \frac{\tau Q E[Q]^2}{2} \right\} \sqrt{\tau Q} \) and noting that \( \tau S = \frac{\tau Q \tau \epsilon}{\tau Q + \tau \epsilon} \), we have

\[
E \left[ U (W) \right] = \frac{k}{\sqrt{\tau [Q | S]}} = \frac{k}{\sqrt{\tau Q + \tau \epsilon}}
\]
Willingness to Pay for Signal

General Problem (No Price Signal)

- Without price signal and signal $S$, agent's expected utility

$$E [U (W)] = \frac{k}{\sqrt{\tau Q}}$$

- If the agent buys a signal at a price of $m_S$ his expected utility is

$$E [U (W - m_S)] = E [- \exp (-\rho (W - m_S))] = E [- \exp (-\rho (W)) \exp (\rho m_S)] = \frac{k}{\sqrt{\tau [Q|S]}} \exp (\rho m_S)$$

- Agent is indifferent when

$$\frac{k}{\sqrt{\tau Q}} = \frac{k}{\sqrt{\tau [Q|S]}} \exp (\rho m_S)$$

- $\Rightarrow$ willingness to pay

$$m_S = \eta \ln \left( \sqrt{\frac{\tau [Q|S]}{\tau Q}} \right)$$

- Willingness to pay depends on the improvement in precision.
Information Acquisition Stage - Grossman-Stiglitz (1980)

- Every agent has to be indifferent between being informed or not.
  cost of signal $c = \eta \ln \left( \sqrt{\frac{\tau[v|S]}{\tau[v|P]}} \right) = \eta \ln \left( \sqrt{\frac{\tau[v+\tau\epsilon]}{\tau[v+\phi\tau\epsilon]}} \right)$
  
  (previous slide)
  This determines $\phi = \frac{\tau\epsilon}{\tau\epsilon + \left( \frac{1}{\lambda^I\eta} \right)^2}$, which in turn pins down $\lambda^I$.

- Comparative Statics (using IFT)
  - $c \uparrow \Rightarrow \phi \downarrow$
  - $\eta \uparrow \Rightarrow \phi \uparrow$ (extreme case: risk-neutrality)
  - $\tau\epsilon \uparrow \Rightarrow \phi \uparrow$
  - $\sigma^2_u \uparrow \Rightarrow \phi \rightarrow$ (number of informed traders $\nearrow$)
  - $\sigma^2_u \downarrow 0 \Rightarrow$ no investor purchases a signal
Information Acquisition Stage

- Further extensions:
  - purchase signals with different precisions (Verrecchia 1982)
  - Optimal sale of information
    - photocopied (newsletter) or individualistic signal (Admati & Pfleiderer)
    - indirect versus direct (Admati & Pfleiderer)
Endogenizing Noise Trader Demand

- endowment shocks or outside opportunity shocks that are correlated with asset
- welfare analysis
  - more private information $\rightarrow$ adverse selection
  - more public information $\rightarrow$ Hirshleifer effect (e.g. genetic testing)
- see papers by Spiegel, Bhattacharya & Rohit, and Vives (2006)
Tips & Tricks

- risk-neutral competitive fringe observing limit order book \( L \)
  \[ p = E[v|L(\cdot)] \]
  - separates risk-sharing from informational aspects