Asset Pricing under Asymmetric Information
Strategic Market Order Models

Markus K. Brunnermeier

Princeton University

August 17, 2007
A Classification of Market Microstructure Models

- simultaneous submission of demand schedules
  - competitive rational expectation models
  - strategic share auctions
- sequential move models
  - screening models in which the market maker submits a supply schedule first
    - static
      - uniform price setting
      - limit order book analysis
  - dynamic sequential trade models with multiple trading rounds
- strategic market order models where the market maker sets prices ex-post
Strategic Market Order Models - Overview

- Kyle (1985) model
  - static version
  - dynamic version (in discrete time)
    - Refresher in Dynamic Programming
  - continuous time version (Back 1992)
- Multi-insider Kyle (1985) version
- Other strategic market order models
Kyle 1985 Model

- **Model Setup**
  - asset return $v \sim \mathcal{N}(p_0, \Sigma_0)$
  - Agents (risk neutral)
    - Insider who knows $v$ and submit market order of size $x$
    - Noise trader who submit market orders of exogenous aggregate size $u \sim \mathcal{N}(0, \sigma_u^2)$
    - Market maker sets competitive price after observing net order flow $X = x + u$
  - Timing (order of moves)
    - Stage 1: Insider & liquidity traders submit market orders
    - Stage 2: Market Maker sets the execution price
  - Repeated trading in dynamic version
### Kyle 1985 Model — Static Version

<table>
<thead>
<tr>
<th><strong>Single informed trader</strong></th>
<th><strong>(Competitive) Market Maker</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0) Information</strong></td>
<td><strong>0) Information</strong></td>
</tr>
<tr>
<td>( v := \text{asset's payoff} )</td>
<td>( X = x + u ) batch net order flow</td>
</tr>
<tr>
<td><strong>1) Conjecture (price-rule)</strong></td>
<td><strong>1) Conjecture (insider trading rule)</strong></td>
</tr>
<tr>
<td>( p = \mu + \lambda(x + u) )</td>
<td>( x = \alpha + \beta v )</td>
</tr>
<tr>
<td><strong>2) No Updating</strong></td>
<td><strong>2) Updating</strong> ( E[v</td>
</tr>
<tr>
<td><strong>3) Optimal Demand</strong></td>
<td><strong>3) Price Setting Rule</strong></td>
</tr>
<tr>
<td>( \max_x E[(v - p)</td>
<td>v]x )</td>
</tr>
<tr>
<td>( \max_x E[v - \mu - \lambda x</td>
<td>v]x )</td>
</tr>
<tr>
<td>FOC: ( x = -\frac{\mu}{2\lambda} + \frac{1}{2\lambda} v )</td>
<td>( p = p_0 + \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} { x + u - \alpha + \beta E[v] } )</td>
</tr>
</tbody>
</table>
| SOC: \( \lambda > 0 \) | |}

| **4) Correct Beliefs** | **4) Correct Beliefs** |
| \( \alpha = -\frac{\mu}{2\lambda}, \beta = \frac{1}{2\lambda} \) | \( \mu = p_0 \) Martingale, \( \lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2} \) |
Kyle 1985 Model — Static Version

- solve for unknown coefficients
  - 4 unknown Greeks
  - 4 equations
- \[ \lambda = \frac{1}{2} \sqrt{\frac{\Sigma_0}{\sigma_u^2}} \]
  - \( \lambda \) (illiquidity) decreases with noise trading, \( \sigma_u^2 \)
  - \( \Sigma_0 \) reflect info advantage of insider
Dynamic Programming - A Refresher

- **The Problem:**
  \[
  \max_{t=1}^{T} \max \left[ \mathbb{E}_t \sum_{s=1}^{T} v_s (\tilde{x}_s, u_s) \right] \quad \forall t \Rightarrow \text{(sequential rationality)}
  \]

  under the following law of motion

  \[
  \tilde{x}_{t+1} = f_t (\tilde{x}_t, u_t, \epsilon_t)
  \]

  \(\tilde{x}_t\): vector of state variables (sufficient state space)
  
  \(u_t\): vector of control variables
  
  \(\epsilon_t\): vector of random shocks

- **Method**
  - Backward Induction
  - Dynamic Programming
Dynamic Programming - A Refresher

- Define Value Function

\[ V_t(x_t) := E_t\left[ \sum_{s=t}^{T} v_s(\tilde{x}_s, u_s^*) \right] \]

- ⇒ Bellman Equation

\[ \max_{u_t} E_t [v_t(x_t, u_t) + V_{t+1}(x_{t+1})] \]

- Start at final date \( T \)

\[ V_{T+1}(\cdot) := 0 \]

⇒ in \( t = T \)

\[ \max_{u_T} E_T [v_T(x_T, u_T)] \]

\[ \text{FOC} \Rightarrow u_T^* = g_T(x_T) \]

⇒ \( V_T(x_T) = E_T[v_T(x_T, u_T^*)] \)
Dynamic Programming - A Refresher

• at date $T - 1$

$$\max_{u_{T-1}, u_T} E_{T-1} \left[ \sum_{s=T-1}^{T} v_s (\tilde{x}_s, u_s) \right]$$

given $V_T (x_T)$

$\Leftrightarrow \max_{u_{T-1}} E_{T-1} [v_{T-1} (x_{T-1}, u_{T-1}) + V_T (x_T)]$

given law of motion

$\Leftrightarrow \max_{u_{T-1}} E_{T-1} [v_{T-1} (x_{T-1}, u_{T-1}) + V_T (f_{T-1} (x_{T-1}, u_{T-1}, \tilde{\varepsilon}_{T-1}))]$

$\Rightarrow u_{T-1}^* = g_{T-1} (x_{T-1})$

$\Rightarrow V_{T-1} = E_{T-1} [v_{T-1} (x_{T-1}, u_{T-1}^*) + V_T (f_{T-1} (x_{T-1}, u_{T-1}^*, \tilde{\varepsilon}_{T-1}))]$

• and so on for date $T - 2$ etc.  (and if they didn't die in the uncertainties they are still solving ...)

• This process is quite time consuming.
Dynamic Programming - A Refresher

Alternative way:

- **Step 1:** “Guess” the general form of the value function

\[ V_{t+1}(x_{t+1}) = H_{t+1}(x_{t+1}) \]

\( e.g. \ H_{t+1}(x_{t+1}) = \alpha_{t+1} x_{t+1}^2 \)

- **Step 2:** Derive optimal level of current control

\[
\max_{u_t} E_t \left[ v_t(x_t, u_t) + H_{t+1}(\tilde{x}_{t+1}) \right]
\]

\[
\max_{u_t} E_t \left[ v_t(x_t, u_t) + H_{t+1}(f_t(x_t, u_t, \varepsilon_t)) \right]
\]

\[ \Rightarrow \ u_t^* = \cdots \]

- **Step 3:** Derive value function and check whether it coincides with general value function

\[ V_t(x_t) = E_t \left[ v_t(x_t, u_t^*) + H_{t+1}(f_t(x_t, u_t^*, \tilde{\varepsilon}_t)) \right] \]

\[ ? \ H_t(x_t) = \alpha_t x_t^2 \]
Kyle (1985) — Dynamic Version

**Insider**

- **Step 1:** Conjectured price setting strategy (pricing rule)

\[
p_n = p_{n-1} + \lambda_n \Delta X_n
\]

\[
= p_{n-1} + \lambda_n (\Delta X_n + \Delta u_n)
\]

\[
\left( \frac{1}{\lambda_t} \approx \text{Liquidity} \right)
\]

- **Step 2:** ‘Guess’ Value function for insider’s profit pricing rule is linear \(\rightarrow\) guess quadratic value function

\[
E[\pi_{n+1} | \tilde{p}_1, \cdots, p_n, \nu] = \alpha_n (\nu - p_n)^2 + \delta_n
\]

Information set up to \(n\)

(expected profit from time \(n + 1\) onwards)

\[
\pi_n = E_n [\pi_{n+1} + (\nu - p_n) \Delta x^i_n]
\]
Kyle (1985) — Dynamic Version

Insider ctd.

- **Step 3:** Write Bellman Equation

\[
\max_{\Delta x_n^i} \mathbb{E} \left[ (v - p_n) \Delta x_n^i + \alpha_n (v - p_n)^2 + \delta_n \mid p_1, \ldots, p_{n-1}, v \right]
\]

- **Step 4:** Given insider's beliefs \( p_n = p_{n-1} + \lambda_n \Delta X_n \)

\[
\max_{\Delta x_n^i} \mathbb{E} \left[ (v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n \Delta u_n) \Delta x_n^i + \alpha_n (v - p_{n-1} - \lambda_n \Delta x_n^i - \lambda_n u_n)^2 + \delta_n \mid I_n \right]
\]

Take expectations

\[
\max_{\Delta x_n^i} \mathbb{E} \left[ (v - p_{n-1} - \lambda_n \Delta x_n^i) \Delta x_n^i + \alpha_n (v - p_{n-1} - \lambda_n \Delta x_n^i)^2 \right]
\]

\[
\hspace{2cm} + \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n
\]

\( u \Rightarrow p_n \text{ noisy} \)
Kyle (1985) — Dynamic Version

**Insider ctd.**

- **Step 5:** maximize

  \[
  \text{FOC: } (v - p_{n-1}) - 2\lambda_n \Delta x_n^i - 2\alpha_n \lambda_n (v - p_{n-1}) + 2\alpha_n \lambda_n^2 \Delta x_n^i = 0
  \]

  \[
  \Delta x_n^i = \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n (1 - \alpha_n \lambda_n)} (v - p_{n-1})
  \]

  \[
  := \beta_n \Delta t_n
  \]

  \[
  \text{SOC: } \lambda_n (1 - \alpha_n \lambda_n) > 0
  \]

- **Step 6:** Check whether ‘guessed’ value fcn is correct

  \[
  E [\pi | I_{n-1}] = \max_{\Delta x_n^i} E \left[ (v - p_n) \Delta x_n^i + \alpha_n (\tilde{v} - \tilde{p}_n)^2 + \delta_n | I_{n-1} \right]
  \]

  \[
  = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1}, \text{ where }
  \]

  \[
  \alpha_{n-1} = \frac{1}{4\lambda_n (1 - \alpha_n \lambda_n)}, \delta_{n-1} = \delta_n + \alpha_n (\lambda_n)^2 \sigma_u^2 \Delta t_n
  \]
Kyle (1985) — Dynamic Version

**Market Maker** (Filtering Problem)

- **Step 1:** MM’s belief about insider’s strategy

\[
\Delta x_n^i = \beta_n \Delta t_n (v - p_{n-1}) \\
\Delta X_n = \beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n
\]

\[
\text{Var}[\Delta u_n] = \sigma_u^2 \Delta t_n
\]

- **Step 2:** MM’s filtering problem

By definition:

\[
p_{n-1} : = E[v|\Delta X_1, \cdots, \Delta X_{n-1}] \\
\Sigma_{n-1} : = \text{Var}[v|\Delta X_1, \cdots, \Delta X_{n-1}]
\]

\[
E[\Delta X_n|\Delta X_1, \cdots, \Delta X_{n-1}] = \beta_n \Delta t_n E[(v - p_{n-1}) + \Delta u_n| \cdots] \\
\text{Var}[\Delta X_n| \cdots] = (\beta_n \Delta t_n)^2 \Sigma_{n-1} + \sigma_u^2 \Delta t_n \\
\text{Cov}[v, \Delta X_n| \cdots] = E[v(\beta_n \Delta t_n (v - p_{n-1}) + \Delta u_n| \cdots]
\]

\[
= \beta_n \Delta t_n \Sigma_{n-1}
\]
Kyle (1985) — Dynamic Version

Now we have all ingredients for the Projection Theorem

\[ p_n = p_{n-1} + \frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2} \Delta X_n \]

\[ \sum_n = V[v|\cdots\Delta X_n] = \Sigma_{n-1} - \frac{(\beta_n \Delta t_n)^2 \Sigma_{n-1}^2}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2} \]

\[ = \frac{\sigma_u^2 \Sigma_{n-1}}{(\beta_n)^2 \Delta t_n \Sigma_{n-1} + \sigma_u^2} \]

\[ \Rightarrow \lambda_n = \frac{\beta_n \Delta t_n \Sigma_{n-1}}{(\beta_n \Delta t_n)^2 \Sigma_{n-1} + \Delta t \sigma_u^2} \]

\[ \sum_n = (1 - \lambda_n \beta_n \Delta t_n) \Sigma_{n-1} = \frac{\sigma_u^2 \lambda_n}{\beta_n} \]

\[ \Rightarrow \lambda_n = \frac{\beta_n \Sigma_n}{\sigma_u^2} \]
Kyle (1985) — Dynamic Version

• **Step 3:** Equate coefficients $\alpha_n, \beta_n, \delta_n, \sum_n$

\[
\begin{align*}
\beta_n \Delta t_n &= \frac{1 - 2\alpha_n \lambda_n}{2\lambda_n(1 - \alpha_n \lambda_n)} \\
\alpha_{n-1} &= \frac{1}{4\lambda_n(1 - \alpha_n \lambda_n)} \\
\delta_{n-1} &= \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n \\
\sum_n &= \sigma_u^2 \sum_{n-1} \\
\lambda_n &= \ldots 
\end{align*}
\] as above

Solve recursive system of equations.

• **Interpretation of Equilibrium**
  • restrain from aggressive trading
    • price impact in current trading round
    • price impact in all future trading rounds
  • ...

Generalizations of Kyle (1985)

- **Multiple Insiders**
  - all have same information
  - all hold different information
  - information is correlated
    \[ \Rightarrow \] see Foster & Viswanathan, JF 51, 1437-1478

- **Risk averse insiders**
  - CARA utility

- etc. etc.