Understanding “Consensus"
The Jouini-Napp Model of Belief Aggregation

Markus Brunnermeier    E. Glen Weyl

Department of Economics
Princeton University

Financial Economics I
Fall 2006
Why study belief aggregation?

- Market prices embed consensus set of beliefs among investors?
- What information do we need to extract beliefs?
- Standard model assumes “market consensus”. What is lost?
- What happens if we use standard model to understand world with heterogeneous beliefs?
- False beliefs, pessimism and first-order risk-aversion
Why study belief aggregation?

- Market prices embed consensus set of beliefs among investors?
- What information do we need to extract beliefs?
- Standard model assumes “market consensus”. What is lost?
- What happens if we use standard model to understand world with heterogeneous beliefs?
- False beliefs, pessimism and first-order risk-aversion
Why study belief aggregation?

- Market prices embed consensus set of beliefs among investors?
- What information do we need to extract beliefs?
- Standard model assumes “market consensus”. What is lost?
- What happens if we use standard model to understand world with heterogeneous beliefs?
- False beliefs, pessimism and first-order risk-aversion

Jouini and Napp (2006)
Why study belief aggregation?

- Market prices embed consensus set of beliefs among investors?
- What information do we need to extract beliefs?
- Standard model assumes “market consensus”. What is lost?
- What happens if we use standard model to understand world with heterogeneous beliefs?
- False beliefs, pessimism and first-order risk-aversion
Why study belief aggregation?

- Market prices embed consensus set of beliefs among investors?
- What information do we need to extract beliefs?
- Standard model assumes “market consensus”. What is lost?
- What happens if we use standard model to understand world with heterogeneous beliefs?
- False beliefs, pessimism and first-order risk-aversion
A simple example of belief aggregation

- Two-periods, two states of the world $\omega_{\text{good}}$ and $\omega_{\text{bad}}$ in the second period.
- $N$ investors with CRRA utility (i.e. $u_i(w) = -e^{-\frac{w}{\theta_i}}$, $\theta_i$ is individual $i$’s constant absolute risk-tolerance), no discounting.
- Complete markets (two securities, each paying off 1 unit of consumption in each state).
- Investors have different beliefs; investor $i$ believes the good state happens with probability $\pi_i \neq \pi_j$.
- Both investors endowed with 2 in the good state, 1 in the bad state and 1.5 today.
A simple example of belief aggregation

- Two-periods, two states of the world $\omega_{\text{good}}$ and $\omega_{\text{bad}}$ in the second period.
- $N$ investors with CRRA utility (i.e. $u_i(w) = -e^{-\frac{w}{\theta_i}}$, $\theta_i$ is individual $i$’s constant absolute risk-tolerance), no discounting.
- Complete markets (two securities, each paying off 1 unit of consumption in each state).
- Investors have different beliefs; investor $i$ believes the good state happens with probability $\pi_i \neq \pi_j$.
- Both investors endowed with 2 in the good state, 1 in the bad state and 1.5 today.
A simple example of belief aggregation

- Two-periods, two states of the world $\omega_{good}$ and $\omega_{bad}$ in the second period.
- $N$ investors with CRRA utility (i.e. $u_i(w) = -e^{-\frac{w}{\theta_i}}$, $\theta_i$ is individual $i$’s constant absolute risk-tolerance), no discounting.
- Complete markets (two securities, each paying off 1 unit of consumption in each state).
- Investors have different beliefs; investor $i$ believes the good state happens with probability $\pi_i \neq \pi_j$.
- Both investors endowed with 2 in the good state, 1 in the bad state and 1.5 today.
A simple example of belief aggregation

- Two-periods, two states of the world $\omega_{\text{good}}$ and $\omega_{\text{bad}}$ in the second period.

- $N$ investors with CRRA utility (i.e. $u_i(w) = -e^{-\frac{w}{\theta_i}}$, $\theta_i$ is individual $i$’s constant absolute risk-tolerance), no discounting.

- Complete markets (two securities, each paying off 1 unit of consumption in each state).

- Investors have different beliefs; investor $i$ believes the good state happens with probability $\pi_i \neq \pi_j$.

- Both investors endowed with 2 in the good state, 1 in the bad state and 1.5 today.
A simple example of belief aggregation

- Two-periods, two states of the world \( \omega_{\text{good}} \) and \( \omega_{\text{bad}} \) in the second period.
- \( N \) investors with CRRA utility (i.e. \( u_i(w) = -e^{-\frac{w}{\theta_i}} \), \( \theta_i \) is individual \( i \)'s constant absolute risk-tolerance), no discounting.
- Complete markets (two securities, each paying off 1 unit of consumption in each state).
- Investors have different beliefs; investor \( i \) believes the good state happens with probability \( \pi_i \neq \pi_j \).
- Both investors endowed with 2 in the good state, 1 in the bad state and 1.5 today.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same. Heterogeneous beliefs $\implies$ heterogeneous endowments.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same. Heterogeneous beliefs $\implies$ heterogeneous endowments.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same.

Heterogeneous beliefs $\Rightarrow$ heterogeneous endowments.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same. Heterogeneous beliefs $\implies$ heterogeneous endowments.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same. Heterogeneous beliefs $\implies$ heterogeneous endowments.
Solve for *equivalent common belief equilibrium* (ECBE). ECBE means:

1. Investors have a common set of beliefs, as in standard model.
2. All prices (i.e. investor marginal valuations and behavior) same as in the original equilibrium.
3. Trading is the same.
4. Aggregate, but not individual, endowment the same. 

Heterogeneous beliefs $\implies$ heterogeneous endowments.
We could solve by brute force:

- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to an ECBE.
- Adjust individual endowments to match the equilibrium.

BUT tedious! Instead, we use a quicker and simpler method.
The old way to solve

We could solve by brute force:
- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to a ECBE.
- Adjust individual endowments to match the equilibrium

BUT tedious! Instead, we use a quicker and simpler method.
The old way to solve

We could solve by brute force:

- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to a ECBE.
- Adjust individual endowments to match the equilibrium

BUT tedious! Instead, we use a quicker and simpler method.
The old way to solve

We could solve by brute force:

- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to a ECBE.
- Adjust individual endowments to match the equilibrium

BUT tedious! Instead, we use a quicker and simpler method.
We could solve by brute force:

- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to a ECBE.
- Adjust individual endowments to match the equilibrium

BUT tedious! Instead, we use a quicker and simpler method.
The old way to solve

We could solve by brute force:

- Solve the consumers’ optimization problem given any prices.
- Find the prices that clear the market.
- Find a probability measure leading to a ECBE.
- Adjust individual endowments to match the equilibrium

BUT tedious! Instead, we use a quicker and simpler method.
An easier way

Directly use condition of consistent marginal valuations.

- Let $y_i^j$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $\bar{y}_i^j$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

\[ \forall i: \pi_i u_i'(y_i^\text{good}) = \pi u_i'(\bar{y}_i^\text{good}) \]
\[ \forall i: (1 - \pi_i) u_i'(y_i^\text{bad}) = (1 - \pi) u_i'(\bar{y}_i^\text{bad}) \]
Directly use condition of consistent marginal valuations.

- Let $y_i^j$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $\bar{y}_i^j$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

$$\forall i : \pi_i u'_i(y_i^{\text{good}}) = \pi u'_i(\bar{y}_i^{\text{good}})$$
$$\forall i : (1 - \pi_i) u'_i(y_i^{\text{bad}}) = (1 - \pi) u'_i(\bar{y}_i^{\text{bad}})$$
An easier way

Directly use condition of consistent marginal valuations.

- Let $y^j_i$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $y^j_i$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

$$\forall i : \pi_i u'_i(y^{good}_i) = \pi u'_i(y^{good}_i)$$

$$\forall i : (1 - \pi_i) u'_i(y^{bad}_i) = (1 - \pi) u'_i(y^{bad}_i)$$

Jouini and Napp (2006)
An easier way

Directly use condition of consistent marginal valuations.

- Let $y^j_i$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $\bar{y}^j_i$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

\[ \forall i : \pi_i u'_i(y_i^{\text{good}}) = \pi u'_i(\bar{y}_i^{\text{good}}) \]
\[ \forall i : (1 - \pi_i) u'_i(y_i^{\text{bad}}) = (1 - \pi) u'_i(\bar{y}_i^{\text{bad}}) \]
An easier way

Directly use condition of consistent marginal valuations.

- Let $y_i^j$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $\tilde{y}_i^j$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

\[ \forall i: \pi_i u_i'(y_i^{\text{good}}) = \bar{\pi} u_i'(\tilde{y}_i^{\text{good}}) \]
\[ \forall i: (1 - \pi_i) u_i'(y_i^{\text{bad}}) = (1 - \bar{\pi}) u_i'(\tilde{y}_i^{\text{bad}}) \]
An easier way

Directly use condition of consistent marginal valuations.

- Let $y^j_i$ be the wealth that $i$ buys in state $j$ in the original equilibrium.
- Let $\bar{y}^j_i$ be the wealth she buys in the ECBE.
- Also, let $\pi$ represent the common belief of the probability of the good state.

$$\forall i : \pi_i u'_i(y^\text{good}_i) = \bar{\pi} u'_i(\bar{y}^\text{good}_i)$$

$$\forall i : (1 - \pi_i) u'_i(y^\text{bad}_i) = (1 - \bar{\pi}) u'_i(\bar{y}^\text{bad}_i)$$
By simple differentiation we have that \( u'_i(x) = \frac{e^{-x/\theta_i}}{\theta_i} \). Property 4 implies aggregate endowment the same. We exploit this:

- Define \( \bar{\theta} \equiv \sum_i \theta_i \).

- We raise the equations for individual \( i \) to the power \( \theta_i/\bar{\theta} \) to put the argument of their marginal utility in “common currency”. We obtain:

\[
\forall i, \pi_i^{\theta_i} e^{-y_i^{\text{good}}/\theta_i} = \bar{\pi}^{\theta_i/\bar{\theta}} e^{-y_i^{\text{good}}/\bar{\theta}}
\]

Corresponding equation for bad state.
By simple differentiation we have that $u'_i(x) = \frac{e^{-x}}{\theta_i}$. Property 4 implies aggregate endowment the same. We exploit this:

- Define $\bar{\theta} \equiv \sum_i \theta_i$.

- We raise the equations for individual $i$ to the power $\frac{\theta_i}{\bar{\theta}}$ to put the argument of their marginal utility in "common currency". We obtain:

$$\forall i, \pi_i^{\frac{\theta_i}{\bar{\theta}}} e^{-\frac{y_{i,\text{good}}}{\bar{\theta}}} = \frac{\theta_i}{\bar{\theta}} \pi_i^{\frac{\theta_i}{\bar{\theta}}} e^{-\frac{y_{i,\text{good}}}{\bar{\theta}}}$$

Corresponding equation for bad state.
By simple differentiation we have that $u'_i(x) = e^{-\frac{x}{\theta_i}}$. Property 4 \[\implies\] aggregate endowment the same. We exploit this:

- Define $\overline{\theta} \equiv \sum_i \theta_i$.

- We raise the equations for individual $i$ to the power $\frac{\theta_i}{\overline{\theta}}$ to put the argument of their marginal utility in "common currency". We obtain:

$$\forall i, \, \pi_i^{\frac{\theta_i}{\overline{\theta}}} e^{-\frac{y_i^{\text{good}}}{\overline{\theta}}} = \overline{\pi}^{\frac{\theta_i}{\overline{\theta}}} e^{-\frac{y_i^{\text{good}}}{\overline{\theta}}}$$

Corresponding equation for bad state.
Solving continued...

Multiplying:

\[
\frac{\theta_1}{\pi_1} \frac{\theta_2}{\pi_2} e^{-\frac{y_{\text{good}} + y_{\text{good}}}{\theta}} = \frac{\bar{y}_{\text{good}} + y_{\text{good}}}{\theta} \]

But aggregate endowments the same:

\[
\pi = \prod_i \frac{\theta_i}{\pi_i}
\]

A corresponding formula \(1 - \pi = \prod_i (1 - \pi_i)^{\frac{\theta_i}{\theta}}\) applies to the bad state.
Multiplying:

\[
\frac{\theta_1}{\pi_1} \frac{\theta_2}{\pi_2} e^{\frac{-y_1^{\text{good}} + y_2^{\text{good}}}{\theta}} = \pi e^{\frac{-y_1^{\text{good}} + y_2^{\text{good}}}{\theta}}
\]

But aggregate endowments the same:

\[
\pi = \prod_i \pi_i^{\theta_i}
\]

A corresponding formula \(1 - \pi = \prod_i (1 - \pi_i)^{\theta_i}\) applies to the bad state.
Multiplying:

\[
\pi_1^{\frac{\theta_1}{\theta}} \pi_2^{\frac{\theta_2}{\theta}} e^{-\frac{y_{\text{good}}^1 + y_{\text{good}}^2}{\theta}} = \pi e^{-\frac{y_{\text{good}}^1 + y_{\text{good}}^2}{\theta}}
\]

But aggregate endowments the same:

\[
\pi = \prod_i \pi_i^{\frac{\theta_i}{\theta}}
\]

A corresponding formula \(1 - \pi = \prod_i (1 - \pi_i)^{\frac{\theta_i}{\theta}}\) applies to the bad state.
Solving continued...

Multiplying:

\[
\frac{\theta_1}{\pi_1} \frac{\theta_2}{\pi_2} \pi \theta_1 \theta_2 \ e^{\frac{-y_1^{\text{good}} + y_2^{\text{good}}}{\theta}} = \pi e^{\frac{-y_1^{\text{good}} + y_2^{\text{good}}}{\theta}}
\]

But aggregate endowments the same:

\[
\pi = \prod_i \pi_i^{\frac{\theta_i}{\theta}}
\]

A corresponding formula \(1 - \pi = \prod_i (1 - \pi_i) \frac{\theta_i}{\theta}\) applies to the bad state.
Note:

- \( \bar{\pi} \) is not a probability measure. Suppose that there are just two investors, \( \theta_1 = \theta_2 = 1 \) and \( \pi_1 = 1 - \pi_2 = .9 \). Then \( \bar{\pi} = 1 - \bar{\pi} = .18 \ll .5 \). Not clear how we should interpret these "common beliefs".

- Effect on common belief proportional to fraction of total risk-bearing; those willing to bet more on their beliefs shape market beliefs more.

- Common belief is weighted geometric mean of the individual beliefs.
Interpreting the solution

Note:

- **$\bar{\pi}$ is not a probability measure.** Suppose that there are just two investors, $\theta_1 = \theta_2 = 1$ and $\pi_1 = 1 - \pi_2 = .9$. Then $\bar{\pi} = 1 - \bar{\pi} = .18 \ll .5$. Not clear how we should interpret these “common beliefs”.

- Effect on common belief proportional to fraction of total risk-bearing; those willing to bet more on their beliefs shape market beliefs more.

- Common belief is weighted geometric mean of the individual beliefs.
Note:

- $\bar{pi}$ is not a probability measure. Suppose that there are just two investors, $\theta_1 = \theta_2 = 1$ and $\pi_1 = 1 - \pi_2 = .9$. Then $\bar{pi} = 1 - \bar{pi} = .18 \ll .5$. Not clear how we should interpret these “common beliefs”.

- Effect on common belief proportional to fraction of total risk-bearing; those willing to bet more on their beliefs shape market beliefs more.

- Common belief is weighted geometric mean of the individual beliefs.
Interpreting the solution

Note:

- \( \bar{\rho_i} \) is not a probability measure. Suppose that there are just two investors, \( \theta_1 = \theta_2 = 1 \) and \( \pi_1 = 1 - \pi_2 = .9 \). Then \( \bar{\rho_i} = 1 - \bar{\rho_i} = .18 \ll .5 \). Not clear how we should interpret these “common beliefs”.

- Effect on common belief proportional to fraction of total risk-bearing; those willing to bet more on their beliefs shape market beliefs more.

- Common belief is weighted geometric mean of the individual beliefs.
What about endowments?

We can also determine how endowments differ in the ECBE:

\[ \pi_i e^{-\frac{y_{i}^{good}}{\theta_i}} = \prod_j \pi_j^{\frac{\theta_j}{\theta_i}} e^{-\frac{y_{i}^{good}}{\theta_i}} \]

Taking the logarithm, arithmetic yield:

\[ \bar{y}_i^{good} - y_i^{good} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] \]

Property 3, \( \implies \)

\[ \bar{e}_i - e_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] = \theta_i \left[ \log(\pi) - \log(\pi_i) \right] \]

Thus, heterogeneous beliefs \( \implies \) heterogeneous endowments.
What about endowments?

We can also determine how endowments differ in the ECBE:

\[
\pi_j e^{-\frac{y_i^{\text{good}}}{\theta_i}} = \prod_j \pi_j^{\frac{\theta_j}{\theta}} e^{-\frac{y_i^{\text{good}}}{\theta_i}}
\]

Taking the logarithm, arithmetic yield:

\[
y_i^{\text{good}} - y_i^{\text{good}} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right]
\]

Property 3, \(\implies\):

\[
\bar{e}_i - e_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] = \theta_i \left[ \log(\bar{\pi}) - \log(\pi_i) \right]
\]

Thus, heterogeneous beliefs \(\implies\) heterogeneous endowments.
What about endowments?

We can also determine how endowments differ in the ECBE:

\[ \pi_j e^{-\frac{y_{i,good}}{\theta_i}} = \prod_j \pi_j^{\theta_j} e^{-\frac{y_{i,good}}{\theta_i}} \]

Taking the logarithm, arithmetic yield:

\[ \bar{y}_{i,good} - y_{i,good} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] \]

Property 3, \( \Rightarrow \):

\[ \bar{e}_i - e_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] = \theta_i \left[ \log(\bar{\pi}) - \log(\pi_i) \right] \]

Thus, heterogeneous beliefs \( \Rightarrow \) heterogeneous endowments.
We can also determine how endowments differ in the ECBE:

\[
\pi_i e^{-\frac{y_i^{good}}{\theta_i}} = \prod_j \pi_j^{\frac{\theta_j}{\theta}} e^{-\frac{y_j^{good}}{\theta_i}}
\]

Taking the logarithm, arithmetic yield:

\[
\bar{y}_i^{good} - y_i^{good} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} [\log(\pi_j) - \log(\pi_i)]
\]

Property 3, \(\Rightarrow\):

\[
\bar{e}_i - e_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} [\log(\pi_j) - \log(\pi_i)] = \theta_i [\log(\bar{\pi}) - \log(\pi_i)]
\]

Thus, heterogeneous beliefs \(\Rightarrow\) heterogeneous endowments.
What about endowments?

We can also determine how endowments differ in the ECBE:

\[ \pi_i e^{-\frac{y_i^{good}}{\theta_i}} = \prod_{j} \pi_j^{\theta_j} e^{-\frac{y_j^{good}}{\theta_j}} \]

Taking the logarithm, arithmetic yield:

\[ y_{i}^{good} - y_{i}^{good} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] \]

Property 3, \(\implies\) :

\[ e_i - \bar{e}_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] = \theta_i \left[ \log(\pi) - \log(\pi_i) \right] \]

Thus, heterogeneous beliefs \(\implies\) heterogeneous endowments.
What about endowments?

We can also determine how endowments differ in the ECBE:

\[
\pi_i e^{-y_i^{\text{good}}} = \prod_j \pi_j^{\theta_j} e^{-y_j^{\text{good}}}/\theta_i
\]

Taking the logarithm, arithmetic yield:

\[
y_i^{\text{good}} - y_i^{\text{good}} = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right]
\]

Property 3, \(\implies\) :

\[
\bar{e}_i - e_i = \theta_i \sum_{j \neq i} \frac{\theta_j}{\theta} \left[ \log(\pi_j) - \log(\pi_i) \right] = \theta_i \left[ \log(\pi) - \log(\pi_i) \right]
\]

Thus, heterogeneous beliefs \(\implies\) heterogeneous endowments.

Jouini and Napp (2006)
Interpreting endowment shift

- Change in belief proportional to: difference from consensus and risk-tolerance.
- If individual has higher probability on a state than the common belief, she tends to trade towards this state.
- Rationalized by individual having a lower endowment in this state (hedging an idiosyncratic risk).
- Thus standard model will interpret trading due to heterogeneous beliefs as resulting from idiosyncratic risks.
Interpreting endowment shift

- Change in belief proportional to: difference from consensus and risk-tolerance.
- If individual has higher probability on a state than the common belief, she tends to trade towards this state.
- Rationalized by individual having a lower endowment in this state (hedging an idiosyncratic risk).
- Thus standard model will interpret trading due to heterogeneous beliefs as resulting from idiosyncratic risks.
Interpreting endowment shift

- Change in belief proportional to: difference from consensus and risk-tolerance.
- If individual has higher probability on a state than the common belief, she tends to trade towards this state.
- Rationalized by individual having a lower endowment in this state (hedging an idiosyncratic risk).
- Thus standard model will interpret trading due to heterogeneous beliefs as resulting from idiosyncratic risks.
Interpreting endowment shift

- Change in belief proportional to: difference from consensus and risk-tolerance.
- If individual has higher probability on a state than the common belief, she tends to trade towards this state.
- Rationalized by individual having a lower endowment in this state (hedging an idiosyncratic risk).
- Thus standard model will interpret trading due to heterogeneous beliefs as resulting from idiosyncratic risks.
Adjustment also proportional to risk tolerance.

Those with large risk tolerance will bet a lot on beliefs: large endowment adjustment necessary to rationalize.

Standard model: risk-tolerant person with idiosyncratic beliefs trading because of large idiosyncratic risks.
Endowments continued...

- Adjustment also proportional to risk tolerance.
- Those with large risk tolerance will bet a lot on beliefs: large endowment adjustment necessary to rationalize
- Standard model: risk-tolerant person with idiosyncratic beliefs trading because of large idiosyncratic risks

Jouini and Napp (2006)
Endowments continued...

- Adjustment also proportional to risk tolerance.
- Those with large risk tolerance will bet a lot on beliefs: large endowment adjustment necessary to rationalize.
- Standard model: risk-tolerant person with idiosyncratic beliefs trading because of large idiosyncratic risks.

Jouini and Napp (2006)
A motivating example of pessimism

Can beliefs generate first-order risk-aversion?

Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
A motivating example of pessimism

Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
A motivating example of pessimism

Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion”

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
A motivating example of pessimism

Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion" 

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0

Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.

Is this plausible? It seems to happen a lot.
Can beliefs generate first-order risk-aversion? Would you consider risk-averse someone who was afraid to fly? This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
A motivating example of pessimism

Can beliefs generate first-order risk-aversion?
Would you consider risk-averse someone who was afraid to fly?
This cannot be (economic) “risk-aversion"

- If arrives earns $u$
- If he drives he earns disutility from inconvenience $v$
- If he crashes he earns utility 0
- Two people, both with the same inconvenience from not flying and value to flying but one chooses not to fly, while the other chooses to fly.
- Is this plausible? It seems to happen a lot.
Risk-aversion is a property of utility functions.

Parameters equal between the two, economic risk-aversion cannot do it. What is?

First individual thinks that crashing is more likely, assigning probability \( q \), second assigns \( p < q \).

Then if \( qu - v > 0 \), but \( pu - v < 0 \) the “risk-averse” individual chooses not to fly, but the “risk-seeking” individual chooses to fly.

Thus “risk-aversion” in standard parlance is about beliefs, not utility. It is a sort of pessimism.

“Luck is out to get me” or “nothing ever goes my way.”

Can we incorporate into finance?

Jouini and Napp (2006) Brunnermeier Slides on Jouini-Napp
Risk-aversion is a property of utility functions.

Parameters equal between the two, economic risk-aversion cannot do it. What is?

First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$

Then if $qu - v > 0$, but $pu - v < 0$ the “risk-averse" individual chooses not to fly, but the “risk-seeking" individual chooses to fly.

Thus “risk-aversion" in standard parlance is about beliefs, not utility. It is a sort of pessimism.

“Luck is out to get me" or “nothing ever goes my way."

Can we incorporate into finance?
Pessimism example explained

- Risk-aversion is a property of utility functions.
- Parameters equal between the two, economic risk-aversion cannot do it. What is?
- First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$
- Then if $qu - v > 0$, but $pu - v < 0$ the "risk-averse" individual chooses not to fly, but the "risk-seeking" individual chooses to fly.
- Thus "risk-aversion" in standard parlance is about beliefs, not utility. It is a sort of pessimism.
- "Luck is out to get me" or "nothing ever goes my way."
- Can we incorporate into finance?

Jouini and Napp (2006)  Brunnermeier Slides on Jouini-Napp
Risk-aversion is a property of utility functions. Parameters equal between the two, economic risk-aversion cannot do it. What is?

First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$

Then if $qu - v > 0$, but $pu - v < 0$ the “risk-averse” individual chooses not to fly, but the “risk-seeking” individual chooses to fly.

Thus “risk-aversion” in standard parlance is about beliefs, not utility. It is a sort of pessimism.

“Luck is out to get me” or “nothing ever goes my way.”

Can we incorporate into finance?
Pessimism example explained

- Risk-aversion is a property of utility functions.
- Parameters equal between the two, economic risk-aversion cannot do it. What is?
- First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$
- Then if $qu - v > 0$, but $pu - v < 0$ the “risk-averse” individual chooses not to fly, but the “risk-seeking” individual chooses to fly.
- Thus “risk-aversion” in standard parlance is about beliefs, not utility. It is a sort of pessimism.
- “Luck is out to get me” or “nothing ever goes my way.”
- Can we incorporate into finance?
Risk-aversion is a property of utility functions.

Parameters equal between the two, economic risk-aversion cannot do it. What is?

First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$

Then if $qu - v > 0$, but $pu - v < 0$ the “risk-averse" individual chooses not to fly, but the “risk-seeking" individual chooses to fly.

Thus “risk-aversion" in standard parlance is about beliefs, not utility. It is a sort of pessimism.

“Luck is out to get me" or “nothing ever goes my way."

Can we incorporate into finance?
Risk-aversion is a property of utility functions.
Parameters equal between the two, economic risk-aversion cannot do it. What is?
First individual thinks that crashing is more likely, assigning probability $q$, second assigns $p < q$
Then if $qu - v > 0$, but $pu - v < 0$ the “risk-averse" individual chooses not to fly, but the “risk-seeking" individual chooses to fly.
Thus “risk-aversion" in standard parlance is about beliefs, not utility. It is a sort of pessimism.
“Luck is out to get me" or “nothing ever goes my way."
Can we incorporate into finance?
A try:

- Suppose objective probability $\pi_\star$ of the good state occurring.
- Individual $i$ "first-order pessimistic" if $\pi_i < \pi_\star$. This definition quite obvious here, but more general later.
- Does pessimism affect the risk-free rate in the same way as risk-aversion? Market price of risk?
A try:

- Suppose objective probability $\pi_\star$ of the good state occurring.
- Individual $i$ “first-order pessimistic” if $\pi_i < \pi_\star$. This definition quite obvious here, but more general later.
- Does pessimism affect the risk-free rate in the same way as risk-aversion? Market price of risk?
A try:

- Suppose objective probability $\pi_\star$ of the good state occurring.
- Individual $i$ “first-order pessimistic" if $\pi_i < \pi_\star$. This definition quite obvious here, but more general later.
- Does pessimism affect the risk-free rate in the same way as risk-aversion? Market price of risk?
A try:

- Suppose objective probability $\pi_\star$ of the good state occurring.
- Individual $i$ “first-order pessimistic" if $\pi_i < \pi_\star$. This definition quite obvious here, but more general later.
- Does pessimism affect the risk-free rate in the same way as risk-aversion? Market price of risk?
Solving the model with pessimism

- Assume there only one investor.
  - Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
  - Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}$.
  - Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.

  - Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}$.
  - Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}} \quad e^{-\frac{3}{2}\theta} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi^*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

- **Assume** there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_\star$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}/\theta$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $\frac{e^{-\frac{3}{2}\theta}}{\theta}$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}\theta$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-2\theta} + (1-\pi)e^{-1\theta}}{\theta}$.

Thus the risk free rate:

$$\frac{\pi e^{-2\theta} + (1-\pi)e^{-1\theta}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.

Jouini and Napp (2006)  Brunnermeier Slides on Jouini-Napp
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $e^{-\frac{3}{2}\theta}$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi_*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
Solving the model with pessimism

• Assume there only one investor.
• Risk-bearing capacity \( \theta \) and belief about the probability of the good state \( \pi \).
• Marginal utility of consumption today is \( \frac{e^{-\frac{3}{2}\theta}}{\theta} \).
• Expected marginal utility tomorrow is \( \frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{\theta} \).

Thus the risk free rate:

\[
\frac{\pi e^{-\frac{2}{\theta}} + (1-\pi)e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2}\theta}} - 1
\]

• \( e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}} \), so risk free rate is clearly decreasing in \( \pi \).
• If \( \pi < \pi_* \) then the risk-free rate is higher than if beliefs are correct.
• Thus pessimism increases risk-free rate.
Solving the model with pessimism

- Assume there only one investor.
- Risk-bearing capacity $\theta$ and belief about the probability of the good state $\pi$.
- Marginal utility of consumption today is $e^{-\frac{3}{2\theta}}$.
- Expected marginal utility tomorrow is $\frac{\pi e^{-\frac{2}{\theta}} + (1 - \pi) e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2\theta}}}$.

Thus the risk free rate:

$$\frac{\pi e^{-\frac{2}{\theta}} + (1 - \pi) e^{-\frac{1}{\theta}}}{e^{-\frac{3}{2\theta}}} - 1$$

- $e^{-\frac{2}{\theta}} < e^{-\frac{1}{\theta}}$, so risk free rate is clearly decreasing in $\pi$.
- If $\pi < \pi^*$ then the risk-free rate is higher than if beliefs are correct.
- Thus pessimism increases risk-free rate.
What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[
\frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}}
\]

And in the bad state:

\[
\frac{(1 - \pi) e^{-\frac{1}{\theta}}}{(1 - \pi^*) e^{-\frac{3}{2\theta}}}
\]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
More on pessimism

What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[ \frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}} \]

And in the bad state:

\[ \frac{(1 - \pi) e^{-\frac{1}{\theta}}}{(1 - \pi^*) e^{-\frac{3}{2\theta}}} \]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.

Jouini and Napp (2006)  Brunnermeier Slides on Jouini-Napp
What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[
\frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}}
\]

And in the bad state:

\[
\frac{(1 - \pi)e^{-\frac{1}{\theta}}}{(1 - \pi^*)e^{-\frac{3}{2\theta}}}
\]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[
\frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}}
\]

And in the bad state:

\[
\frac{(1 - \pi) e^{-\frac{1}{\theta}}}{(1 - \pi^*) e^{-\frac{3}{2\theta}}}
\]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
More on pessimism

What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[
\frac{\pi e^{-\frac{2}{\theta}}}{\pi_* e^{-\frac{3}{2\theta}}}
\]

And in the bad state:

\[
\frac{(1 - \pi) e^{-\frac{1}{\theta}}}{(1 - \pi_*) e^{-\frac{3}{2\theta}}}
\]

- Kernel in good state increasing in \(\pi\).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[ \frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}} \]

And in the bad state:

\[ \frac{(1 - \pi)e^{-\frac{1}{\theta}}}{(1 - \pi^*)e^{-\frac{3}{2\theta}}} \]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
More on pessimism

What effect does pessimism have on the pricing kernel under objective measure? The value (objective) value of the pricing kernel in the good state is:

\[
\frac{\pi e^{-\frac{2}{\theta}}}{\pi^* e^{-\frac{3}{2\theta}}}
\]

And in the bad state:

\[
\frac{(1 - \pi) e^{-\frac{1}{\theta}}}{(1 - \pi^*) e^{-\frac{3}{2\theta}}}
\]

- Kernel in good state increasing in \( \pi \).
- Kernel in bad state is decreasing.
- Pessimism decreases the kernel in the good state, increases it in the bad state.
Interpreting pessimism

- Leads to a greater “separation” of the price kernel for the two states. This is an increase in the “risk-premium”, Thus, similar effect on market prices to risk aversion.
- Optimism has opposite effect; in fact, enough optimism can lead the good state kernel to be higher than the low-state kernel, i.e. risk-seeking.
- Are there differences?
Interpreting pessimism

- Leads to a greater “separation” of the price kernel for the two states. This is an increase in the “risk-premium”, Thus, similar effect on market prices to risk aversion.

- Optimism has opposite effect; in fact, enough optimism can lead the good state kernel to be higher than the low-state kernel, i.e. risk-seeking.

- Are there differences?
Interpreting pessimism

- Leads to a greater “separation" of the price kernel for the two states. This is an increase in the “risk-premium", Thus, similar effect on market prices to risk aversion.

- Optimism has opposite effect; in fact, enough optimism can lead the good state kernel to be higher than the low-state kernel, i.e. risk-seeking.

- Are there differences?

Jouini and Napp (2006)
A problem with pessimism

Suppose two individuals

Each has utility $u(c) = \log c$

If coin comes up heads, first endowed with 18, second with 2

If tails, first endowed with 9, second with 1.

Suppose fair coin. Person 1 knows this; person 2 is drastically pessimistic: believes that probability is .9 that comes up tails.
A problem with pessimism

- Suppose two individuals
- Each has utility $u(c) = \log c$
  - If coin comes up heads, first endowed with 18, second with 2
  - If tails, first endowed with 9, second with 1.
- Suppose fair coin. Person 1 knows this; person 2 is drastically pessimistic: believes that probability is .9 that comes up tails.
A problem with pessimism

- Suppose two individuals
- Each has utility $u(c) = \log c$
- If coin comes up heads, first endowed with 18, second with 2
- If tails, first endowed with 9, second with 1.

Suppose fair coin. Person 1 knows this; person 2 is drastically pessimistic: believes that probability is .9 that comes up tails.
Suppose two individuals
Each has utility $u(c) = \log c$
If coin comes up heads, first endowed with 18, second with 2
If tails, first endowed with 9, second with 1.
Suppose fair coin. Person 1 knows this; person 2 is drastically pessimistic: believes that probability is .9 that comes up tails.
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
  - First individual spends \( \frac{1}{2} \) of his wealth on consumption in good state; has \( \frac{9}{10} \) of total perfectly correlated wealth.
  - Second spends \( \frac{1}{10} \) and has \( \frac{1}{10} \).
- Let \( w \equiv 20 + 10p_2 \), where \( p_{tails} \) is price of consumption in the tails state.
- Market clearing implies \( \left( \frac{9}{20} + \frac{1}{100} \right)w = 20 \).
- \( w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23} \)
- Investor 2 buys \( c_2^1 = \frac{100}{207} \approx \frac{1}{2} \) and \( c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1 \).
- Investor 1 buys \( c_1^1 = \frac{450}{23} \approx 19\frac{1}{2} \) and \( c_1^2 = 6\frac{1}{2} > \frac{3}{10} c_1^1 \)
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends $\frac{1}{2}$ of his wealth on consumption in good state; has $\frac{9}{10}$ of total perfectly correlated wealth.

- Second spends $\frac{1}{10}$ and has $\frac{1}{10}$.
- Let $w = 20 + 10p_2$, where $p_{tails}$ is price of consumption in the tails state.
- Market clearing implies $\left(\frac{9}{20} + \frac{1}{100}\right)w = 20$.

- $w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23}$
- Investor 2 buys $c_2^1 = \frac{100}{207} \approx \frac{1}{2}$ and $c_2^2 = \frac{5}{3} = \frac{10}{3}c_2^1$.
- Investor 1 buys $c_1^1 = \frac{450}{23} \approx 19\frac{1}{2}$ and $c_1^2 = 6\frac{1}{2} > \frac{3}{10}c_1^1$
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends $\frac{1}{2}$ of his wealth on consumption in good state; has $\frac{9}{10}$ of total perfectly correlated wealth.
- Second spends $\frac{1}{10}$ and has $\frac{1}{10}$.
- Let $w \equiv 20 + 10p_2$, where $p_{tails}$ is price of consumption in the tails state.
- Market clearing implies $\left(\frac{9}{20} + \frac{1}{100}\right)w = 20$.
- $w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23}$
- Investor 2 buys $c_2^1 = \frac{100}{207} \approx \frac{1}{2}$ and $c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1$.
- Investor 1 buys $c_1^1 = \frac{450}{23} \approx 19 \frac{1}{2}$ and $c_1^2 = 6 \frac{1}{2} > \frac{3}{10} c_1^1$
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends $\frac{1}{2}$ of his wealth on consumption in good state; has $\frac{9}{10}$ of total perfectly correlated wealth.
- Second spends $\frac{1}{10}$ and has $\frac{1}{10}$.
- Let $w \equiv 20 + 10p_2$, where $p_{tails}$ is price of consumption in the tails state.

- Market clearing implies $\left(\frac{9}{20} + \frac{1}{100}\right)w = 20$.
- $w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23}$
- Investor 2 buys $c_2^1 = \frac{100}{207} \approx \frac{1}{2}$ and $c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1$.
- Investor 1 buys $c_1^1 = \frac{450}{23} \approx 19\frac{1}{2}$ and $c_1^2 = 6\frac{1}{2} > \frac{3}{10} c_1^1$. 

Jouini and Napp (2006)
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends \( \frac{1}{2} \) of his wealth on consumption in good state; has \( \frac{9}{10} \) of total perfectly correlated wealth.
- Second spends \( \frac{1}{10} \) and has \( \frac{1}{10} \).
- Let \( w \equiv 20 + 10p_2 \), where \( p_{\text{tails}} \) is price of consumption in the tails state.
- Market clearing implies \( \left( \frac{9}{20} + \frac{1}{100} \right) w = 20 \).
  - \( w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23} \)
  - Investor 2 buys \( c_2^1 = \frac{100}{207} \approx \frac{1}{2} \) and \( c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1 \).
  - Investor 1 buys \( c_1^1 = \frac{450}{23} \approx 19 \frac{1}{2} \) and \( c_1^2 = 6 \frac{1}{2} > \frac{3}{10} c_1^1 \).
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends $\frac{1}{2}$ of his wealth on consumption in good state; has $\frac{9}{10}$ of total perfectly correlated wealth.
- Second spends $\frac{1}{10}$ and has $\frac{1}{10}$.
- Let $w \equiv 20 + 10p_2$, where $p_{\text{tails}}$ is price of consumption in the tails state.
- Market clearing implies $\left(\frac{9}{20} + \frac{1}{100}\right) w = 20$.
- $w = \frac{1000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23}$
- Investor 2 buys $c_2^1 = \frac{100}{207} \approx \frac{1}{2}$ and $c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1$.
- Investor 1 buys $c_1^1 = \frac{450}{23} \approx 19\frac{1}{2}$ and $c_1^2 = 6\frac{1}{2} > \frac{3}{10} c_1^1$
Normalize price of consumption in good state to 1.

First individual spends \( \frac{1}{2} \) of his wealth on consumption in good state; has \( \frac{9}{10} \) of total perfectly correlated wealth.

Second spends \( \frac{1}{10} \) and has \( \frac{1}{10} \).

Let \( w \equiv 20 + 10p_2 \), where \( p_{tails} \) is price of consumption in the tails state.

Market clearing implies \( \left( \frac{9}{20} + \frac{1}{100} \right) w = 20 \).

\[
\begin{align*}
    w &= \frac{1000}{23} = 20 + 10p_2 \\
    &\implies p_2 = \frac{54}{23} \\
    \text{Investor 2 buys} &\quad c_2^1 = \frac{100}{207} \approx \frac{1}{2} \quad \text{and} \quad c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1.
\end{align*}
\]

\[
\begin{align*}
    \text{Investor 1 buys} &\quad c_1^1 = \frac{450}{23} \approx 19 \frac{1}{2} \quad \text{and} \quad c_1^2 = 6 \frac{1}{2} > \frac{3}{10} c_1^1.
\end{align*}
\]
Solving the problematic pessimism example

- Normalize price of consumption in good state to 1.
- First individual spends \( \frac{1}{2} \) of his wealth on consumption in good state; has \( \frac{9}{10} \) of total perfectly correlated wealth.
- Second spends \( \frac{1}{10} \) and has \( \frac{1}{10} \).
- Let \( w \equiv 20 + 10p_2 \), where \( p_{tails} \) is price of consumption in the tails state.
- Market clearing implies \( \left( \frac{9}{20} + \frac{1}{100} \right) w = 20 \).
- \( w = \frac{10000}{23} = 20 + 10p_2 \implies p_2 = \frac{54}{23} \)
- Investor 2 buys \( c_2^1 = \frac{100}{207} \approx \frac{1}{2} \) and \( c_2^2 = \frac{5}{3} = \frac{10}{3} c_2^1 \).
- Investor 1 buys \( c_1^1 = \frac{450}{23} \approx 19 \frac{1}{2} \) and \( c_1^2 = 6 \frac{1}{2} > \frac{3}{10} c_1^1 \).
Thus pessimistic individual has less balanced portfolio!

Very different from risk aversion...bets on bad outcomes.

Is this interesting?

Alternatively, could have individuals that believe things go against them. Might generate behavior more like risk aversion, but like non-expected utility.
What does this show us about pessimism?

- Thus pessimistic individual has less balanced portfolio!
- Very different from risk aversion...bets on bad outcomes.
- Is this interesting?
- Alternatively, could have individuals that believe things go against them. Might generate behavior more like risk aversion, but like non-expected utility.
What does this show us about pessimism?

- Thus pessimistic individual has less balanced portfolio!
- Very different from risk aversion...bets on bad outcomes.
- Is this interesting?

- Alternatively, could have individuals that believe things go against them. Might generate behavior more like risk aversion, but like non-expected utility.
Thus pessimistic individual has less balanced portfolio!

Very different from risk aversion...bets on bad outcomes.

Is this interesting?

Alternatively, could have individuals that believe things go against them. Might generate behavior more like risk aversion, but like non-expected utility.
How does it generalize?

These were nice toy examples. But how do the insights generalize? How do we define concepts in a fully general rigorous manner?
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e^i_t$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^*^i$
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e_t^i$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^*^i$
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e_t^i$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^*_i$
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e^i_t$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^{*i}$
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e_t^i$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^{*i}$
Assumptions of model

- $N$ agents, $T$ periods
- Differentiable, concave, increasing utility
- Endowments $e^i_t$ bounded below w.p.1 and above
- Individual probability $Q^i$
- Complete markets: optimal individual consumption $y^*_i$
We look for ECBE with common “characteristic" $M$, endowments $\bar{e^i}$ and consumption $\bar{y^i}$ such that:

1. $Q_t u^i_t(t, \bar{y}^i) = Q_t u^i_t(t, \bar{y}^*i) \quad \forall t, i$
2. $y^*i - e^i = \bar{y}^i - \bar{e}^i \quad \forall i$

Can be shown that this always exists and is unique.
We look for ECBE with common "characteristic" $M$, endowments $\bar{e}^i$ and consumption $\bar{y}^i$ such that:

1. $Q_t u'_i(t, \bar{y}^i) = Q_t u'_i(t, y^*_i)$ $\forall t, i$
2. $y^*_i - e^i = \bar{y}^i - \bar{e}^i$ $\forall i$

Can be shown that this always exists and is unique.
We look for ECBE with common “characteristic" $M$, endowments $\bar{e}^i$ and consumption $\bar{y}^i$ such that:

1. $Q_t u'_i(t, \bar{y}^i) = Q_t u'_i(t, y^*_i) \quad \forall t, i$
2. $y^*_i - e^i = \bar{y}^i - \bar{e}^i \quad \forall i$

Can be shown that this always exists and is unique.
We look for ECBE with common “characteristic" $M$, endowments $\bar{e}^i$ and consumption $\bar{y}^i$ such that:

1. $Q_t u'_i(t, \bar{y}^i) = Q'_t u'_i(t, y'^i) \quad \forall t, i$
2. $y'^i - e^i = \bar{y}^i - \bar{e}^i \quad \forall i$

Can be shown that this always exists and is unique.
Already derived formula with exponential utility. A generalization:

1. Suppose utility HARA with common slope:
   \[ \frac{u'}{u''} = \theta_i + \eta x \]
2. Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
3. Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q^i)^\eta \right]^{\frac{1}{\eta}}
\]
Already derived formula with exponential utility. A generalization:

- Suppose utility HARA with common slope:
  \[ \frac{u'}{u''} = \theta_i + \eta x \]
- Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
- Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q^i)^\eta \right]^{\frac{1}{\eta}}
\]
Already derived formula with exponential utility. A generalization:

- Suppose utility HARA with common slope:
  \[ \frac{u'}{u''} = \theta_i + \eta x \]
- Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
- Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q^i)^\eta \right]^{\frac{1}{\eta}}
\]
Already derived formula with exponential utility. A generalization:

- Suppose utility HARA with common slope:
  \[ \frac{u'}{u''} = \theta_i + \eta x \]
- Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
- Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q^i)^{\eta} \right]^{\frac{1}{\eta}}
\]
Already derived formula with exponential utility. A generalization:

- Suppose utility HARA with common slope:
  \[ \frac{u'}{u''} = \theta_i + \eta x \]
- Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
- Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q^i)^\eta \right]^{\frac{1}{\eta}}
\]
Already derived formula with exponential utility. A generalization:

- Suppose utility HARA with common slope:
  \[ \frac{u'}{u''} = \theta_i + \eta x \]
- Let \( \lambda_i \) be the lagrange multiplier in \( i \)'s max
- Let \( \gamma_i \equiv \frac{\lambda_i^{-\eta}}{\sum_j \lambda_j^{-\eta}} \)

Then \( Q \) is given by:

\[
Q = \left[ \sum_i \gamma_i (Q_i)^{\eta} \right]^{\frac{1}{\eta}}
\]
Interpretation

- **Weighted “mean” of beliefs**
  - Weight proportional to wealth \((\lambda^{-\eta})\) as this determines risk bearing
  - Closer to geometric mean as \(\eta \to 0\) (exponential case), arithmetic mean if \(\eta = 1\), “superarithmetic” if \(\eta > 1\)
  - Therefore \(\eta < 1 \implies Q(\Omega) < 1\), \(\eta = 1 \implies Q(\Omega = 1)\), \(\eta > 1 \implies Q(\Omega) > 1\)
  - This is aggregation “bias”, cumulates over time
  - Define \(B_t\) by \(B_0 = 1\), \(B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}\); then \(B\) is a sort of deflator
  - Then \(\overline{Q} \equiv \frac{Q}{B}\) is a belief
Interpretation

- Weighted “mean” of beliefs
- Weight proportional to wealth \((\lambda^{-\eta})\) as this determines risk bearing
  - Closer to geometric mean as \(\eta \rightarrow 0\) (exponential case), arithmetic mean if \(\eta = 1\), “superarithmetic” if \(\eta > 1\)
  - Therefore \(\eta < 1 \implies Q(\Omega) < 1\), \(\eta = 1 \implies Q(\Omega = 1)\), \(\eta > 1 \implies Q(\Omega) > 1\)
- This is aggregation “bias”, cumulates over time
- Define \(B_t\) by \(B_0 = 1\), \(B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}\); then \(B\) is a sort of deflator
- Then \(\bar{Q} = \frac{Q}{B}\) is a belief
Interpretation

- Weighted "mean" of beliefs
- Weight proportional to wealth ($\lambda^{-\eta}$) as this determines risk bearing
- Closer to geometric mean as $\eta \to 0$ (exponential case), arithmetic mean if $\eta = 1$, "superarithmetic" if $\eta > 1$
- Therefore $\eta < 1 \implies Q(\Omega) < 1$, $\eta = 1 \implies Q(\Omega = 1)$, $\eta > 1 \implies Q(\Omega) > 1$
- This is aggregation "bias", cumulates over time
- Define $B_t$ by $B_0 = 1$, $B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}$; then $B$ is a sort of deflator
- Then $\bar{Q} \equiv \frac{Q}{B}$ is a belief
Interpretation

- Weighted “mean” of beliefs
- Weight proportional to wealth \((\lambda^{-\eta})\) as this determines risk bearing
- Closer to geometric mean as \(\eta \to 0\) (exponential case), arithmetic mean if \(\eta = 1\), “superarithmetic” if \(\eta > 1\)
- Therefore \(\eta < 1 \implies Q(\Omega) < 1\), \(\eta = 1 \implies Q(\Omega = 1)\), \(\eta > 1 \implies Q(\Omega) > 1\)
- This is aggregation “bias”, cumulates over time
- Define \(B_t\) by \(B_0 = 1\), \(B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}\); then \(B\) is a sort of deflator
- Then \(\overline{Q} \equiv \frac{Q}{B}\) is a belief
Weighted “mean” of beliefs

Weight proportional to wealth ($\lambda^{-\eta}$) as this determines risk bearing

Closer to geometric mean as $\eta \to 0$ (exponential case), arithmetic mean if $\eta = 1$, “superarithmetic” if $\eta > 1$

Therefore $\eta < 1 \implies Q(\Omega) < 1$, $\eta = 1 \implies Q(\Omega = 1)$, $\eta > 1 \implies Q(\Omega) > 1$

This is aggregation “bias”, cumulates over time

Define $B_t$ by $B_0 = 1$, $B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}$; then $B$ is a sort of deflator

Then $\overline{Q} \equiv \frac{Q}{B}$ is a belief
Interpretation

- Weighted “mean” of beliefs
- Weight proportional to wealth ($\lambda^{-\eta}$) as this determines risk bearing
- Closer to geometric mean as $\eta \to 0$ (exponential case), arithmetic mean if $\eta = 1$, “superarithmetic” if $\eta > 1$
- Therefore $\eta < 1 \implies Q(\Omega) < 1$, $\eta = 1 \implies Q(\Omega = 1)$, $\eta > 1 \implies Q(\Omega) > 1$
- This is aggregation “bias”, cumulates over time
- Define $B_t$ by $B_0 = 1$, $B_t = B_{t-1} \frac{E_t[Q_t]}{Q_{t-1}}$; then $B$ is a sort of deflator
- Then $\overline{Q} \equiv \frac{Q}{B}$ is a belief
Interpretation

- Weighted “mean” of beliefs
- Weight proportional to wealth ($\lambda^{-\eta}$) as this determines risk bearing
- Closer to geometric mean as $\eta \to 0$ (exponential case), arithmetic mean if $\eta = 1$, “superarithmetic” if $\eta > 1$
- Therefore $\eta < 1 \implies Q(\Omega) < 1$, $\eta = 1 \implies Q(\Omega = 1)$, $\eta > 1 \implies Q(\Omega) > 1$
- This is aggregation “bias”, cumulates over time
- Define $B_t$ by $B_0 = 1$, $B_t = B_{t-1} \frac{E_{t-1}[Q_t]}{Q_{t-1}}$; then $B$ is a sort of deflator
- Then $\frac{Q}{B}$ is a belief
Only skimmed over asset pricing in our example. Let’s solve it generally here:

- **Objective measure** $P$
- **Riskless asset** $S^0$ with rate $r_{t+1}^f \equiv \frac{S_{t+1}^0}{S_t^0} - 1$ and risky assets with prices $S_t^i$ and return $R_{t+1}^i \equiv \frac{S_{t+1}^i}{S_t^i} - 1$
- If $\pi^*$ is equilibrium price deflator process (under $P$) then $\pi^* S$ is martingale
- Thus $E_t^P[R_{t+1}] - r_{t+1}^f = -\text{cov}_t^P \left[ \frac{\pi_{t+1}^*}{E_t^P[\pi_{t+1}^*]}, R_{t+1} \right]$
Setting up the CCAPM

Only skimmed over asset pricing in our example. Let’s solve it generally here:

- **Objective measure** \( P \)
  - Riskless asset \( S^0 \) with rate \( r^f_{t+1} \equiv \frac{S^0_{t+1}}{S^0_{t}} - 1 \) and risky assets with prices \( S^i_t \) and return \( R^i_{t+1} \equiv \frac{S^i_{t+1}}{S^i_{t}} - 1 \)
  - If \( \pi^* \) is equilibrium price deflator process (under \( P \)) then \( \pi^* S \) is martingale
  - Thus \( E_t^P [R_{t+1}] - r^f_{t+1} = -\text{cov}_t^P \left[ \frac{\pi^*_{t+1}}{E_t^P [\pi^*_{t+1}]}, R_{t+1} \right] \)
Only skimmed over asset pricing in our example. Let’s solve it generally here:

- **Objective measure** $P$
- **Riskless asset** $S^0$ with rate $r_{t+1}^f \equiv \frac{S_{t+1}^0}{S_t^0} - 1$ and risky assets with prices $S^i_t$ and return $R_{t+1}^i \equiv \frac{S_{t+1}^i}{S_t^i} - 1$
- If $\pi^*$ is equilibrium price deflator process (under $P$) then $\pi^* S$ is martingale
- Thus $E_t^P[R_{t+1}] - r_{t+1}^f = -\text{cov}_t^P\left[\frac{\pi_{t+1}^*}{E_t^P[\pi_{t+1}^*]}, R_{t+1}\right]$
Setting up the CCAPM

Only skimmed over asset pricing in our example. Let’s solve it generally here:

- **Objective measure** $P$
- **Riskless asset** $S^0$ with rate $r_{t+1}^f \equiv \frac{S^0_{t+1}}{S^0_t} - 1$ and risky assets with prices $S^i_t$ and return $R^i_{t+1} \equiv \frac{S^i_{t+1}}{S^i_t} - 1$
- If $\pi^*$ is equilibrium price deflator process (under $P$) then $\pi^* S$ is martingale

Thus $E_t^P [R_{t+1}] - r_{t+1}^f = -\text{cov}_t^P \left[ \frac{\pi^*_{t+1}}{E_t^P [\pi^*_{t+1}]}, R_{t+1} \right]$
Setting up the CCAPM

Only skimmed over asset pricing in our example. Let’s solve it generally here:

- Objective measure $P$
- Riskless asset $S^0$ with rate $r_{t+1}^f \equiv \frac{S_{t+1}^0}{S_t^0} - 1$ and risky assets with prices $S_t^i$ and return $R_{t+1}^i \equiv \frac{S_{t+1}^i}{S_t^i} - 1$
- If $\pi^*$ is equilibrium price deflator process (under $P$) then $\pi^* S$ is martingale
- Thus $E_t^P[R_{t+1}] - r_{t+1}^f = -\text{cov}_t^P \left[ \frac{\pi_{t+1}^*}{E_t^P[\pi_{t+1}^*]}, R_{t+1} \right]$