Lecture 03: One Period Model: Pricing

Prof. Markus K. Brunnermeier
Overview: Pricing

1. LOOP, No arbitrage
2. Parity relationship between options
3. No arbitrage and existence of state prices
4. Market completeness and uniqueness of state prices
5. Pricing kernel \( q^* \)
6. Four pricing formulas (state prices, SDF, EMM, state pricing)
7. Recovering state prices from options
Vector Notation

- Notation: $y, x \in \mathbb{R}^n$
  - $y \geq x \iff y^i \geq x^i$ for each $i=1,\ldots,n$.
  - $y > x \iff y \geq x$ and $y \neq x$.
  - $y >> x \iff y^i > x^i$ for each $i=1,\ldots,n$.

- Inner product
  - $y \cdot x = \sum_i yx$

- Matrix multiplication
Three Forms of No-ARBITRAGE

1. Law of one price (LOOP)
   If \( h'X = k'X \) then \( p \cdot h = p \cdot k \).

2. No strong arbitrage
   There exists no portfolio \( h \) which is a strong arbitrage, that is \( h'X \geq 0 \) and \( p \cdot h < 0 \).

3. No arbitrage
   There exists no strong arbitrage nor portfolio \( k \) with \( k'X > 0 \) and \( p \cdot k \leq 0 \).
Three Forms of No-ARBITRAGE

- Law of one price is equivalent to every portfolio with zero payoff has zero price.
- No arbitrage $\Rightarrow$ no strong arbitrage
  No strong arbitrage $\Rightarrow$ law of one price
Overview: Pricing

1. LOOP, No arbitrage
2. Forwards
3. Parity relationship between options
4. No arbitrage and existence of state prices
5. Market completeness and uniqueness of state prices
6. Pricing kernel $q^*$
7. Four pricing formulas (state prices, SDF, EMM, state pricing)
8. Recovering state prices from options
Alternative ways to buy a stock

- Four different payment and receipt timing combinations:
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later
  - Forward contract: agree on price now, pay/receive later

- Payments, receipts, and their timing:

<table>
<thead>
<tr>
<th>Description</th>
<th>Pay at Time:</th>
<th>Receive Security at Time:</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outright Purchase</td>
<td>0</td>
<td>0</td>
<td>$S_0$ at time 0</td>
</tr>
<tr>
<td>Fully Leveraged Purchase</td>
<td>$T$</td>
<td>0</td>
<td>$S_0e^{rT}$ at time $T$</td>
</tr>
<tr>
<td>Prepaid Forward Contract</td>
<td>0</td>
<td>$T$</td>
<td>?</td>
</tr>
<tr>
<td>Forward Contract</td>
<td>$T$</td>
<td>$T$</td>
<td>$? \times e^{rT}$</td>
</tr>
</tbody>
</table>
Pricing prepaid forwards

• If we can price the prepaid forward ($F^{P}$), then we can calculate the price for a forward contract:

$$F = \text{Future value of } F^{P}$$

• Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant
  - Price of the prepaid forward contract same as current stock price
  - $F_{0, T}^{P} = S_{0}$ (where the asset is bought at $t = 0$, delivered at $t = T$)

• Pricing by discounted preset value ($\alpha$: risk-adjusted discount rate)
  - If expected $t=T$ stock price at $t=0$ is $E_{0}(S_{T})$, then $F_{0, T}^{P} = E_{0}(S_{T}) e^{-\alpha T}$
  - Since $t=0$ expected value of price at $t=T$ is $E_{0}(S_{T}) = S_{0} e^{\alpha T}$
  - Combining the two, $F_{0, T}^{P} = S_{0} e^{\alpha T} e^{-\alpha T} = S_{0}$
Pricing prepaid forwards (cont.)

- Pricing by arbitrage

  - If at time $t=0$, the prepaid forward price somehow exceeded the stock price, i.e., $F_{0,T}^p > S_0$, an arbitrageur could do the following:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Stock @ $S_0$</td>
<td>Time 0: $-S_0$, Time $T$: $+S_T$</td>
</tr>
<tr>
<td>Sell Prepaid Forward @ $F_{0,T}^p$</td>
<td>$+F_{0,T}^p$, $-S_T$</td>
</tr>
<tr>
<td>Total</td>
<td>$F_{0,T}^p$, $S_0$, 0</td>
</tr>
</tbody>
</table>

Table 5.2: Cash flows and transactions to undertake arbitrage when the prepaid forward price, $F_{0,T}^p$, exceeds the stock price, $S_0$. 
Pricing prepaid forwards (cont.)

- What if there are deterministic* dividends? Is $F^P_{0, T} = S_0$ still valid?
  - No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock $\Rightarrow F^P_{0, T} < S_0$

$$F^P_{0, T} = S_0 - PV(\text{all dividends paid from } t=0 \text{ to } t=T)$$

- For discrete dividends $D_{t_i}$ at times $t_i, i = 1, \ldots, n$
  - The prepaid forward price: $F^P_{0, T} = S_0 - \sum_{i=1}^{n} PV_{0, t_i}(D_{t_i})$

- For continuous dividends with an annualized yield $\delta$
  - The prepaid forward price: $F^P_{0, T} = S_0 e^{-\delta T}$

*NB: if dividends are stochastic, we cannot apply the one period model
Pricing prepaid forwards (cont.)

- Example 5.1
  - XYZ stock costs $100 today and will pay a quarterly dividend of $1.25. If the risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?
  - \[ F^P_{0,1} = 100 - \sum_{i=1}^{4} 1.25e^{-0.025i} = 95.30 \]

- Example 5.2
  - The index is $125 and the dividend yield is 3% continuously compounded. How much does a 1-year prepaid forward cost?
  - \[ F^P_{0,1} = 125e^{-0.03} = 121.31 \]
Pricing forwards on stock

• Forward price is the future value of the prepaid forward
  - No dividends: \( F_{0,T} = FV(FP_{0,T}) = FV(S_0) = S_0 e^{rT} \)
  - Discrete dividends: \( F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)}D_{t_i} \)
  - Continuous dividends: \( F_{0,T} = S_0 e^{(r-\delta)T} \)

• Forward premium
  - The difference between current forward price and stock price
  - Can be used to infer the current stock price from forward price
  - Definition:
    • Forward premium = \( F_{0,T} / S_0 \)
    • Annualized forward premium =: \( \pi^a = (1/T) \ln (F_{0,T} / S_0) \) (from \( e^{\pi T} = F_{0,T} / S_0 \))
Creating a synthetic forward

- One can offset the risk of a forward by creating a synthetic forward to offset a position in the actual forward contract.
- How can one do this? (assume continuous dividends at rate $\delta$)
  - Recall the long forward payoff at expiration: $S_T - F_{0,T}$
  - Borrow and purchase shares as follows:

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy $e^{-\delta T}$ Units of the Index</td>
<td>Time 0</td>
</tr>
<tr>
<td>Borrow $S_0 e^{-\delta T}$</td>
<td>$-S_0 e^{-\delta T}$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>

- Note that the total payoff at expiration is same as forward payoff.
Creating a *synthetic forward* (cont.)

- The idea of creating synthetic forward leads to following:
  - Forward = Stock – zero-coupon bond
  - Stock = Forward + zero-coupon bond
  - Zero-coupon bond = Stock – forward

- Cash-and-carry arbitrage: Buy the index, short the forward

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transaction</strong></td>
<td><strong>Time 0</strong></td>
</tr>
<tr>
<td>Buy Tailed Position in Stock, Paying $S_0e^{-\delta T}$</td>
<td>$-S_0e^{-\delta T}$</td>
</tr>
<tr>
<td>Borrow $S_0e^{-\delta T}$</td>
<td>$+S_0e^{-\delta T}$</td>
</tr>
<tr>
<td>Short Forward</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>
Other issues in forward pricing

• Does the forward price predict the future price?
  - According the formula $F_{0,T} = S_0 e^{(r-\delta)T}$ the forward price conveys no additional information beyond what $S_0$, $r$, and $\delta$ provides
  - Moreover, the forward price underestimates the future stock price

• Forward pricing formula and cost of carry
  - Forward price =
    \[
    \text{Spot price} + \text{Interest to carry the asset} - \text{asset lease rate}
    \]
    Cost of carry, $(r-\delta)S$
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Put-Call Parity

• For European options with the same strike price and time to expiration the parity relationship is:

$$\text{Call} - \text{put} = PV(\text{forward price} - \text{strike price})$$

or

$$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

• Intuition:

Buying a call and selling a put with the strike equal to the forward price ($F_{0,T} = K$) creates a synthetic forward contract and hence must have a zero price.
Parity for Options on Stocks

• If underlying asset is a stock and Div is the deterministic* dividend stream, then $e^{-rT} F_{0,T} = S_0 - PV_{0,T}(Div)$, therefore

$$C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}(K)$$

• Rewriting above,

$$S_0 = C(K, T) - P(K, T) + PV_{0,T}(Div) + e^{-rT}(K)$$

• For index options, $S_0 - PV_{0,T}(Div) = S_0 e^{-\delta T}$, therefore

$$C(K, T) = P(K, T) + S_0 e^{-\delta T} - PV_{0,T}(K)$$

*allows to stick with one period setting
Properties of option prices

• American vs. European

Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option:

\[ C_{\text{Amer}}(S, K, T) \geq C_{\text{Eur}}(S, K, T) \]

\[ P_{\text{Amer}}(S, K, T) \geq P_{\text{Eur}}(S, K, T) \]

• Option price boundaries

Call price cannot:
- be negative,
- exceed stock price,
- be less than price implied by put-call parity using zero for put price:

\[ S > C_{\text{Amer}}(S, K, T) \geq C_{\text{Eur}}(S, K, T) > \max [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)] \]
Properties of option prices (cont.)

- Option price boundaries
  - Call price cannot:
    - be negative
    - exceed stock price
    - be less than price implied by put-call parity using zero for put price:
      \[ S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)] \]
  - Put price cannot:
    - be more than the strike price
    - be less than price implied by put-call parity using zero for call price:
      \[ K > P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max [0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})] \]
Properties of option prices (cont.)

- Early exercise of American options
  - A non-dividend paying American call option should not be exercised early, because:
    \[ C_{\text{Amer}} \geq C_{\text{Eur}} \geq S_t - K + P_{\text{Eur}} + K(1 - e^{-r(T-t)}) > S_t - K \]
  - That means, one would lose money by exercising early instead of selling the option
  - If there are dividends, it may be optimal to exercise early
  - It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low
Properties of option prices (cont.)

- Time to expiration
  - An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  - European call options on dividend-paying stock and European puts may be less valuable than an otherwise identical option with less time to expiration.
  - A European call option on a non-dividend paying stock will be more valuable than an otherwise identical option with less time to expiration.
  - When the strike price grows at the rate of interest, European call and put prices on a non-dividend paying stock increases with time.
    - Suppose to the contrary $P(T) < P(t)$ for $T > t$, then arbitrage. Buy $P(T)$ and sell $P(t)$. At $t$ if $S_t > K_t$, $P(t) = 0$, if $S_t < K_t$, payoff $S_t - K_t$. Keep stock and finance $K_t$. Time $T$-value $S_T - K_t e^{r(T-t)} = S_T - K_T$. 

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Properties of option prices (cont.)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price:
    \[ C(K_1) \geq C(K_2) \]
  - A put with a high strike price is at least as valuable as an otherwise identical call with low strike price:
    \[ P(K_2) \geq P(K_1) \]
  - The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:
    \[ C(K_1) - C(K_2) \leq K_2 - K_1 \]
Properties of option prices (cont.)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices:
    $$P(K_1) - P(K_2) \leq K_2 - K_1$$
  - Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price):
    $$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$
Properties of option prices (cont.)

The example in Panel A violates the proposition that the rate of change of the option premium must decrease as the strike price rises. The rate of change from 50 to 59 is 5.1/9, while the rate of change from 59 to 65 is 3.9/6. We can arbitrage this convexity violation with an asymmetric butterfly spread. Panel B shows that we earn at least $3 plus interest at time $T$.

### Panel A

<table>
<thead>
<tr>
<th>Strike</th>
<th>50</th>
<th>59</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Premium</td>
<td>14</td>
<td>8.9</td>
<td>5</td>
</tr>
</tbody>
</table>

### Panel B

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Time 0</th>
<th>$S_T &lt; 50$</th>
<th>$50 \leq S_T \leq 59$</th>
<th>$59 \leq S_T \leq 65$</th>
<th>$S_T &gt; 65$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Four 50-</td>
<td>-56</td>
<td>0</td>
<td>$4(S_T - 50)$</td>
<td>$4(S_T - 50)$</td>
<td>$4(S_T - 50)$</td>
</tr>
<tr>
<td>Strike Calls</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sell Ten 59</td>
<td>89</td>
<td>0</td>
<td>0</td>
<td>$10(59 - S_T)$</td>
<td>$10(59 - S_T)$</td>
</tr>
<tr>
<td>Strike Calls</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy Six 65-</td>
<td>-30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$6(S_T - 65)$</td>
</tr>
<tr>
<td>Strike Calls</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lend $3$</td>
<td>-3</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
<td>$3e^{rT}$</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>$3e^{rT}$</td>
<td>$3e^{rT} + 4(S_T - 50)$</td>
<td>$3e^{rT} + 6(65 - S_T)$</td>
<td>$3e^{rT}$</td>
</tr>
</tbody>
</table>

Slide 2-25
Properties of option prices (cont.)

![Graph showing the relationship between option premium and strike price for calls and puts.](#)
Summary of parity relationships

<table>
<thead>
<tr>
<th>Underlying Asset</th>
<th>Parity Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures Contract</td>
<td>( e^{-rT} F_{0,T} = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
<tr>
<td>Stock, No-Dividend</td>
<td>( S_0 = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
<tr>
<td>Stock, Discrete Dividend</td>
<td>( S_0 - PV_{0,T}(Div) = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
<tr>
<td>Stock, Continuous Dividend</td>
<td>( e^{-\delta T} S_0 = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
<tr>
<td>Currency</td>
<td>( e^{-rT} x_0 = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
<tr>
<td>Bond</td>
<td>( B_0 - PV_{0,T}(Coupons) = C(K, T) - P(K, T) + e^{-rT} K )</td>
</tr>
</tbody>
</table>

Versions of put-call parity. Notation in the table includes the spot currency exchange rate, \( x_0 \); the risk-free interest rate in the foreign currency, \( r_f \); and the current bond price, \( B_0 \).
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... back to the big picture

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities

**Problem:**
- *Difficult to observe risk preferences*
- *What can we say about existence of state prices without assuming specific utility functions/constraints for all agents in the economy*
Vector Notation

- Notation: $y, x \in \mathbb{R}^n$
  - $y \geq x \iff y_i \geq x_i$ for each $i = 1, \ldots, n.$
  - $y > x \iff y \geq x$ and $y \neq x.$
  - $y >> x \iff y_i > x_i$ for each $i = 1, \ldots, n.$

- Inner product
  - $y \cdot x = \sum_i y_i x_i$

- Matrix multiplication
Three Forms of No-ARBITRAGE

1. **Law of one price (LOOP)**
   If $h'X = k'X$ then $p \cdot h = p \cdot k$.

2. **No strong arbitrage**
   There exists no portfolio $h$ which is a strong arbitrage, that is $h'X \geq 0$ and $p \cdot h < 0$.

3. **No arbitrage**
   There exists no strong arbitrage nor portfolio $k$ with $k'X > 0$ and $p \cdot k \leq 0$. 
Three Forms of No-ARBITRAGE

- Law of one price is equivalent to every portfolio with zero payoff has zero price.
- No arbitrage $\Rightarrow$ no strong arbitrage
  No strong arbitrage $\Rightarrow$ law of one price
Define for each $z \in \langle X \rangle$,

$$q(z) := \{ p \cdot h : z = h'X \}$$

If LOOP holds $q(z)$ is a single-valued and linear functional. (i.e. if $h'$ and $h'$ lead to same $z$, then price has to be the same)

Conversely, if $q$ is a linear functional defined in $\langle X \rangle$ then the law of one price holds.
Pricing

• LOOP ⇒ \( q(h'X) = p \cdot h \)

• A linear functional \( Q \) in \( R^S \) is a valuation function if \( Q(z) = q(z) \) for each \( z \in <X> \).

• \( Q(z) = q \cdot z \) for some \( q \in R^S \), where \( q^s = Q(e_s) \), and \( e_s \) is the vector with \( e_s^s = 1 \) and \( e_s^i = 0 \) if \( i \neq s \)

  \( e_s \) is an Arrow-Debreu security

• \( q \) is a vector of state prices
State prices $q$

- $q$ is a vector of state prices if $p = Xq$, that is $p^j = x^j \cdot q$ for each $j = 1, ..., J$

- If $Q(z) = q \cdot z$ is a valuation functional then $q$ is a vector of state prices

- Suppose $q$ is a vector of state prices and LOOP holds. Then if $z = h'X$ LOOP implies that
  
  $$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x^j_s q_s) h^j = \sum_s (\sum_j x^j_s h^j) q_s = q \cdot z$$

- $Q(z) = q \cdot z$ is a valuation functional $\iff$ $q$ is a vector of state prices and LOOP holds
State prices $q$

\[
p(1,1) = q_1 + q_2
\]
\[
p(2,1) = 2q_1 + q_2
\]

Value of portfolio $(1,2)$

\[
3p(1,1) - p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2 = q_1 + 2q_2
\]
The Fundamental Theorem of Finance

• **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with $q >> 0$.

• **Proposition 1'**. Let $X$ be an $J \times S$ matrix, and $p \in R^J$. There is no $h$ in $R^J$ satisfying $h \cdot p \leq 0$, $h'X \geq 0$ and at least one strict inequality if, and only if, there exists a vector $q \in R^S$ with $q >> 0$ and $p = Xq$.

No arbitrage $\Leftrightarrow$ positive state prices
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Multiple State Prices \( q \) & Incomplete Markets

What state prices are consistent with \( p(1,1) \)?

\[
p(1,1) = q_1 + q_2
\]

One equation – two unknowns \( q_1, q_2 \)

There are (infinitely) many.

e.g. if \( p(1,1) = 0.9 \)

\[
q_1 = 0.45, \quad q_2 = 0.45
\]

or \( q_1 = 0.35, \quad q_2 = 0.55 \)
complete markets
Fin 501: Asset Pricing

\[ p = Xq \]

Q(x)

incomplete markets

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Fin 501: Asset Pricing

One Period Model

Q(x)

p = Xq^o

<x>
incomplete markets

x_1

x_2

q^o
Multiple q in incomplete markets

Many possible state price vectors s.t. \( p = X'q \).
One is special: \( q^* \) - it can be replicated as a portfolio.
Uniqueness and Completeness

• **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.

• If markets are not complete, then there exists \( v \in R^S \) with \( 0 = Xv \).

Suppose there is no arbitrage and let \( q >> 0 \) be a vector of state prices. Then \( q + \alpha \ v >> 0 \) provided \( \alpha \) is small enough, and \( p = X \ (q + \alpha \ v) \). Hence, there are an infinite number of strictly positive state prices.
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Four Asset Pricing Formulas

1. State prices
   \[ p^j = \sum_s q_s x_s^j \]

2. Stochastic discount factor
   \[ p^j = E[mx^j] \]

3. Martingale measure
   \[ p^j = \frac{1}{1+r^f} E_{\pi} [x^j] \]
   (reflect risk aversion by over(under)weighing the “bad(good)” states!)

4. State-price beta model
   \[ E[R^j] - R^f = \beta^j E[R^* - R^f] \]
   (in returns \( R^j := \frac{x^j}{p^j} \))
1. State Price Model

- ... so far price in terms of Arrow-Debreu (state) prices

\[ p^j = \sum_s q_s x^j_s \]
2. Stochastic Discount Factor

\[ p^j = \sum_s q_s x^j_s = \sum_s \pi_s \frac{q_s}{\pi_s} x^j_s \]

- That is, stochastic discount factor \( m_s = \frac{q_s}{\pi_s} \) for all \( s \).

\[ p^j = E[ m x^j ] \]
2. Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_S}$
Risk-adjustment in payoffs

\[ p = E[mx^i] = E[m]E[x] + \text{Cov}[m,x] \]

Since \( 1 = E[mR] \), the risk free rate is \( R^f = 1/E[m] \)

\[ p = E[x]/R^f + \text{Cov}[m,x] \]

Remarks:
(i) If risk-free rate does not exist, \( R^f \) is the shadow risk free rate
(ii) In general \( \text{Cov}[m,x] < 0 \), which lowers price and increases return
3. Equivalent Martingale Measure

- Price of any asset
  \[ p^j = \sum_s q_s x^j_s \]

- Price of a bond
  \[ p_{\text{bond}} = \sum_s q_s = \frac{1}{1+r_f} \]

\[ p^j = \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x^j_s \]

\[ p^j = \frac{1}{1+r_f} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x^j_s \]

\[ p^j = \frac{1}{1+r_f} E_{\hat{\pi}} [x^j] \]
... in Returns: \( R_j = x_j / p_j \)

\[
E[m R_j] = 1 \quad \quad \quad \quad R^f \quad E[m] = 1 \\
\Rightarrow E[m(R_j - R^f)] = 0 \\
E[m] \{E[R_j] - R^f\} + \text{Cov}[m, R_j] = 0
\]

\[
E[R_j] - R^f = - \frac{\text{Cov}[m, R_j]}{E[m]} \quad (2)
\]

also holds for portfolios \( h \)

**Note:**

- risk correction depends only on Cov of payoff/return with discount factor.
- Only compensated for taking on systematic risk not idiosyncratic risk.
4. State-price BETA Model

Let the underlying asset be \( x = (1.2, 1) \).

Shrink axes by factor \( \sqrt{\pi_s} \).

\[ R^* = \alpha \, m^* \]

\( p = 1 \) (priced with \( m^* \)).
4. State-price BETA Model

\[ E[R^j] - R^f = - \frac{\text{Cov}[m, R^j]}{E[m]} \] (2)
also holds for all portfolios \( h \) and we can replace \( m \) with \( m^* \).

Suppose (i) \( \text{Var}[m^*] > 0 \) and (ii) \( R^* = \alpha m^* \) with \( \alpha > 0 \)

\[ E[R^h] - R^f = - \frac{\text{Cov}[R^*, R^h]}{E[R^*]} \] (2')

Define \( \beta^h := \frac{\text{Cov}[R^*, R^h]}{\text{Var}[R^*]} \) for any portfolio \( h \)
4. State-price BETA Model

\( (2) \text{ for } R^*: E[R^*] - R^f = -\frac{\text{Cov}[R^*, R^*]}{E[R^*]} \)
\( = -\frac{\text{Var}[R^*]}{E[R^*]} \)

\( (2) \text{ for } R^h: E[R^h] - R^f = -\frac{\text{Cov}[R^*, R^h]}{E[R^*]} \)
\( = -\beta^h \frac{\text{Var}[R^*]}{E[R^*]} \)

\[ E[R^h] - R^f = \beta^h E[R^* - R^f] \]

where \( \beta^h := \frac{\text{Cov}[R^*, R^h]}{\text{Var}[R^*]} \)

very general – but what is \( R^* \) in reality?

Regression \( R^h_s = \alpha^h + \beta^h (R^*)_s + \varepsilon_s \) with \( \text{Cov}[R^*, \varepsilon] = E[\varepsilon] = 0 \)
Four Asset Pricing Formulas

1. State prices
   \[ 1 = \sum_s q_s R_s^j \]

2. Stochastic discount factor
   \[ 1 = E[mR_j] \]

3. Martingale measure
   \[ 1 = 1/(1+r^f) E_{\pi}^\wedge [R_j] \]
   (reflect risk aversion by
   over(under)weighing the “bad(good)” states!)

4. State-price beta model
   \[ E[R_j] - R^f = \beta^j E[R^* - R^f] \]
   (in returns \( R_j := x_j/p_j \))
What do we know about $q$, $m$, $\hat{\pi}$, $R^*$?

- Main results so far
  - Existence iff no arbitrage
  - Hence, single factor only

- but doesn’t famos Fama-French factor model has 3 factors?

  - multiple factor is due to time-variation
    (wait for multi-period model)

- Uniqueness if markets are complete
Different Asset Pricing Models

\[ p_t = E[m_{t+1} x_{t+1}] \]  \[ \Rightarrow \]  \[ E[R^h] - R^f = \beta^h E[R^* - R^f] \]

where \( \beta^h := \frac{\text{Cov}[R^*, R^h]}{\text{Var}[R^*]} \)

\[ f(\cdot) = \text{asset pricing model} \]

**General Equilibrium**

\[ f(\cdot) = \frac{\text{MRS}}{\pi} \]

**Factor Pricing Model**

\[ a + b_1 f_{1,t+1} + b_2 f_{2,t+1} \]

**CAPM**

\[ a + b_1 f_{1,t+1} = a + b_1 R^M \]

Is \( b_1 < 0? \)
Different Asset Pricing Models

• Theory
  - All economics and modeling is determined by
    $$m_{t+1} = a + b' f$$
  - Entire content of model lies in restriction of SDF

• Empirics
  - $m^*$ (which is a portfolio payoff) prices as well as $m$
    (which is e.g. a function of income, investment etc.)
  - measurement error of $m^*$ is smaller than for any $m$
  - Run regression on *returns* (portfolio payoffs)!
    (e.g. Fama-French three factor model)
Fin 501: Asset Pricing

specify Preferences & Technology

observe/specify existing Asset Prices

• evolution of states
• risk preferences
• aggregation

NAC/LOOP

State Prices q
(or stochastic discount factor/Martingale measure)

NAC/LOOP

absolute asset pricing

derive Asset Prices

relative asset pricing

derive Price for (new) asset

Only works as long as market completeness doesn’t change

10:59 Lecture 02

One Period Model

Slide 2-60
Overview: Pricing - one period model

1. LOOP, No arbitrage
2. Forwards
3. Parity relationship between options
4. No arbitrage and existence of state prices
5. Market completeness and uniqueness of state prices
6. Pricing kernel $q^*$
7. Four pricing formulas (state prices, SDF, EMM, beta-pricing)
8. Recovering state prices from options
Recovering State Prices from Option Prices

- Suppose that $S_T$, the price of the underlying portfolio (we may think of it as a proxy for price of “market portfolio”), assumes a "continuum" of possible values.
- Suppose there are a “continuum” of call options with different strike/exercise prices $\Rightarrow$ markets are complete.
- Let us construct the following portfolio:
  for some small positive number $\varepsilon > 0$,
  - Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$
  - Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$
  - Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$
Recovering State Prices ... (ctd.)

Figure 8-2   Payoff Diagram: Portfolio of Options

Value of the portfolio at expiration
Let us thus consider buying $\frac{1}{\epsilon}$ units of the portfolio. The total payment, when $\hat{S}_T - \frac{\delta}{2} \leq S_T \leq \hat{S}_T + \frac{\delta}{2}$, is $\epsilon \cdot \frac{1}{\epsilon} = 1$, for any choice of $\epsilon$. We want to let $\epsilon \mapsto 0$, so as to eliminate the payments in the ranges $S_T \in (\hat{S}_T - \frac{\delta}{2} - \epsilon, \hat{S}_T - \frac{\delta}{2})$ and $S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \epsilon)$. The value of $\frac{1}{\epsilon}$ units of this portfolio is:

$$\frac{1}{\epsilon} \left[ C(S, E = \hat{S}_T - \frac{\delta}{2} - \epsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2}) - C(S, E = \hat{S}_T + \frac{\delta}{2} + \epsilon) \right] \right]$$
Taking the limit $\varepsilon \to 0$

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2}) \right] \right\}
\]

\[
= -\lim_{\varepsilon \to 0} \left\{ \frac{C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2})}{-\varepsilon} \right\} + \lim_{\varepsilon \to 0} \left\{ \frac{C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2})}{\varepsilon} \right\}
\]

\[
\leq 0
\]

\[
= -\frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2}) + \frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2})
\]

Payoff

Divide by $\delta$ and let $\delta \to 0$ to obtain state price density as $\partial^2 C / \partial E^2$.
Recovering State Prices … (ctd.)

Evaluating following cash flow

\[
\text{CF}_T = \begin{cases} 
0 \text{ if } S_T \notin \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right], \\
50000 \text{ if } S_T \in \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right].
\end{cases}
\]

The value today of this cash flow is:

\[
50000 \left[ \frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2}) - \frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2}) \right]
\]

\[
q(S_T^1, S_T^2) = \frac{\partial C}{\partial E}(S, E = S_T^2) - \frac{\partial C}{\partial E}(S, E = S_T^1)
\]
### Table 8.1 Pricing an Arrow-Debreu State Claim

<table>
<thead>
<tr>
<th>E</th>
<th>C(S,E)</th>
<th>Cost of position</th>
<th>Payoff if $S_T =$</th>
<th>ΔC</th>
<th>Δ(ΔC) = $q_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3.354</td>
<td></td>
<td></td>
<td></td>
<td>-0.895</td>
</tr>
<tr>
<td>8</td>
<td>2.459</td>
<td></td>
<td></td>
<td></td>
<td>0.106</td>
</tr>
<tr>
<td>9</td>
<td>1.670</td>
<td>+1.670</td>
<td>0 0 0 1 2 3 4</td>
<td></td>
<td>-0.789</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.164</td>
</tr>
<tr>
<td>10</td>
<td>1.045</td>
<td>-2.090</td>
<td>0 0 0 0 -2 -4 -6</td>
<td></td>
<td>-0.625</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.184</td>
</tr>
<tr>
<td>11</td>
<td>0.604</td>
<td>+0.604</td>
<td>0 0 0 0 0 1 2</td>
<td></td>
<td>-0.441</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.162</td>
</tr>
<tr>
<td>12</td>
<td>0.325</td>
<td></td>
<td></td>
<td></td>
<td>-0.279</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.118</td>
</tr>
<tr>
<td>13</td>
<td>0.164</td>
<td></td>
<td></td>
<td></td>
<td>-0.161</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.184</td>
</tr>
</tbody>
</table>
specify Preferences & Technology

- evolution of states
- risk preferences
- aggregation

observe/specify existing Asset Prices

State Prices \( q \)
(or stochastic discount factor/Martingale measure)

Absolute asset pricing

derive Asset Prices

only works as long as market completeness doesn’t change
The following is for later lecture
Futures contracts

• Exchange-traded “forward contracts”
• Typical features of futures contracts
  - Standardized, with specified delivery dates, locations, procedures
  - A clearinghouse
    • Matches buy and sell orders
    • Keeps track of members’ obligations and payments
    • After matching the trades, becomes counterparty
• Differences from forward contracts
  - Settled daily through the mark-to-market process ➔ low credit risk
  - Highly liquid ➔ easier to offset an existing position
  - Highly standardized structure ➔ harder to customize
Example: S&P 500 Futures

- WSJ listing:

<table>
<thead>
<tr>
<th></th>
<th>OPEN</th>
<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHANGE</th>
<th>HIGH</th>
<th>LOW</th>
<th>INT.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S&amp;P 500 Index (CME)-$250 times index</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>112350</td>
<td>112370</td>
<td>109100</td>
<td>109530</td>
<td>-2810</td>
<td>134960</td>
<td>94100</td>
<td>474,811</td>
</tr>
<tr>
<td>June</td>
<td>111950</td>
<td>111950</td>
<td>109350</td>
<td>109730</td>
<td>-2830</td>
<td>170550</td>
<td>95030</td>
<td>17,224</td>
</tr>
<tr>
<td>Dec</td>
<td>111580</td>
<td>111580</td>
<td>110020</td>
<td>110390</td>
<td>-2930</td>
<td>150070</td>
<td>96130</td>
<td>304</td>
</tr>
<tr>
<td>Est vol</td>
<td>79,914</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>vol Fri</td>
<td>65,250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>open int</td>
<td>502,626</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Idx prf:</td>
<td>Hi 1122.20</td>
<td>Lo 1092.25</td>
<td>Close 1094.44</td>
<td>-27.76</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Contract specifications:

<table>
<thead>
<tr>
<th>Specifications for the S&amp;P 500 index futures contract.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Underlying</strong></td>
</tr>
<tr>
<td><strong>Where traded</strong></td>
</tr>
<tr>
<td><strong>Size</strong></td>
</tr>
<tr>
<td><strong>Months</strong></td>
</tr>
<tr>
<td><strong>Trading ends</strong></td>
</tr>
<tr>
<td><strong>Settlement</strong></td>
</tr>
</tbody>
</table>
Example: S&P 500 Futures (cont.)

- Notional value: $250 x Index
- Cash-settled contract
- Open interest: total number of buy/sell pairs
- Margin and mark-to-market
  - Initial margin
  - Maintenance margin (70-80% of initial margin)
  - Margin call
  - Daily mark-to-market
- Futures prices vs. forward prices
  - The difference negligible especially for short-lived contracts
  - Can be significant for long-lived contracts and/or when interest rates are correlated with the price of the underlying asset
Example: S&P 500 Futures (cont.)

- Mark-to-market proceeds and margin balance for 8 long futures:

**TABLE 5.8**

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier ($)</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000.00</td>
<td>1100.00</td>
<td>—</td>
<td>220,000.00</td>
</tr>
<tr>
<td>1</td>
<td>2000.00</td>
<td>1027.99</td>
<td>-72.01</td>
<td>76,233.99</td>
</tr>
<tr>
<td>2</td>
<td>2000.00</td>
<td>1037.88</td>
<td>9.89</td>
<td>96,102.01</td>
</tr>
<tr>
<td>3</td>
<td>2000.00</td>
<td>1073.23</td>
<td>35.35</td>
<td>166,912.96</td>
</tr>
<tr>
<td>4</td>
<td>2000.00</td>
<td>1048.78</td>
<td>-24.45</td>
<td>118,205.66</td>
</tr>
<tr>
<td>5</td>
<td>2000.00</td>
<td>1090.32</td>
<td>41.54</td>
<td>201,422.13</td>
</tr>
<tr>
<td>6</td>
<td>2000.00</td>
<td>1106.94</td>
<td>16.62</td>
<td>234,894.67</td>
</tr>
<tr>
<td>7</td>
<td>2000.00</td>
<td>1110.98</td>
<td>4.04</td>
<td>243,245.86</td>
</tr>
<tr>
<td>8</td>
<td>2000.00</td>
<td>1024.74</td>
<td>-86.24</td>
<td>71,046.69</td>
</tr>
<tr>
<td>9</td>
<td>2000.00</td>
<td>1007.30</td>
<td>-17.44</td>
<td>36,248.72</td>
</tr>
<tr>
<td>10</td>
<td>2000.00</td>
<td>1011.65</td>
<td>4.35</td>
<td>44,990.57</td>
</tr>
</tbody>
</table>

Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The final row represents expiration of the contract.
Example: S&P 500 Futures (cont.)

- S&P index arbitrage: comparison of formula prices with actual prices:

<table>
<thead>
<tr>
<th>Expiration Month:</th>
<th>March</th>
<th>June</th>
<th>December</th>
</tr>
</thead>
<tbody>
<tr>
<td>Days to Expiration:</td>
<td>42</td>
<td>140</td>
<td>322</td>
</tr>
<tr>
<td>S&amp;P 500 Index Futures Price</td>
<td>1130.4</td>
<td>1132.5</td>
<td>1140.3</td>
</tr>
<tr>
<td>Treasury-Bill Yield</td>
<td>0.0167</td>
<td>0.017</td>
<td>0.0218</td>
</tr>
<tr>
<td>Theoretical Forward Price</td>
<td>1130.68</td>
<td>1131.93</td>
<td>1139.01</td>
</tr>
<tr>
<td>LIBOR</td>
<td>0.0187</td>
<td>0.0201</td>
<td>0.0240</td>
</tr>
<tr>
<td>Theoretical Forward Price</td>
<td>1130.94</td>
<td>1133.28</td>
<td>1141.22</td>
</tr>
</tbody>
</table>

S&P 500 index futures prices and interest rate information from the *Wall Street Journal*, February 1, 2002. The closing S&P 500 spot price was 1130.20. Treasury-bill yields are reported yields on Treasury bills expiring in the same month as the futures contract. LIBOR rates are constructed from Eurodollar prices. The theoretical forward prices are constructed for each maturity from equation (5.7) using the interest rate in the preceding row and assuming a 1.3% dividend yield.
Uses of index futures

- Why buy an index futures contract instead of synthesizing it using the stocks in the index? Lower transaction costs
- Asset allocation: switching investments among asset classes
- Example: Invested in the S&P 500 index and temporarily wish to temporarily invest in bonds instead of index. What to do?
  - Alternative #1: Sell all 500 stocks and invest in bonds
  - Alternative #2: Take a short forward position in S&P 500 index

**TABLE 5.10**

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Today</td>
</tr>
<tr>
<td>Own Stock @ $100</td>
<td>$-100</td>
</tr>
<tr>
<td>Short Forward @ $110</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$-100</td>
</tr>
</tbody>
</table>
Uses of index futures (cont.)

- $100 million portfolio with $\beta$ of 1.4 and $r_f = 6\%$

1. Adjust for difference in $\$ amount
   - 1 futures contract $\$250 \times 1100 = \$275,000$
   - Number of contracts needed $\frac{100\text{mill}}{0.275\text{mill}} = 363.636$

2. Adjust for difference in $\beta$
   - $363.636 \times 1.4 = 509.09$ contracts
Uses of index futures (cont.)

- Cross-hedging with perfect correlation

<table>
<thead>
<tr>
<th>S&amp;P 500 Index</th>
<th>Gain on 509 Futures</th>
<th>Portfolio Value</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>33.855</td>
<td>72.145</td>
<td>106.000</td>
</tr>
<tr>
<td>950</td>
<td>27.491</td>
<td>78.509</td>
<td>106.000</td>
</tr>
<tr>
<td>1000</td>
<td>21.127</td>
<td>84.873</td>
<td>106.000</td>
</tr>
<tr>
<td>1050</td>
<td>14.764</td>
<td>91.236</td>
<td>106.000</td>
</tr>
<tr>
<td>1100</td>
<td>8.400</td>
<td>97.600</td>
<td>106.000</td>
</tr>
<tr>
<td>1150</td>
<td>2.036</td>
<td>103.964</td>
<td>106.000</td>
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<tr>
<td>1200</td>
<td>-4.327</td>
<td>110.327</td>
<td>106.000</td>
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</table>

- Cross-hedging with imperfect correlation
- General asset allocation: futures overlay
- Risk management for stock-pickers
Currency contracts

- Widely used to hedge against changes in exchange rates
- WSJ listing:

<table>
<thead>
<tr>
<th>CURRENCY</th>
<th>OPEN</th>
<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHANGE</th>
<th>LIFETIME HIGH</th>
<th>LIFETIME LOW</th>
<th>OPEN INT.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Japan Yen (CME) - 12.5 million yen; $ per yen (.00)</td>
<td>.7528</td>
<td>.7600</td>
<td>.7506</td>
<td>.7576</td>
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<td>.7547</td>
<td>.7611</td>
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<tr>
<td>Canadian Dollar (CME) - 100,000 drs.; $ per Can $</td>
<td>.6292</td>
<td>.6276</td>
<td>.6264</td>
<td>.6266</td>
<td>− .0016</td>
<td>.6725</td>
<td>.6170</td>
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<td>.6280</td>
<td>.6287</td>
<td>.6283</td>
<td>.6264</td>
<td>− .0016</td>
<td>.6700</td>
<td>.6180</td>
<td>9,652</td>
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<td>.6265</td>
<td>.6282</td>
<td>.6265</td>
<td>.6266</td>
<td>− .0016</td>
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<td>.6269</td>
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<td>− .0016</td>
<td>.6555</td>
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<td>1.4206</td>
<td>1.4108</td>
<td>1.4186</td>
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<td>1.4700</td>
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<td>1.4140</td>
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<td>1.4102</td>
<td>+ .0058</td>
<td>1.4550</td>
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<td>Swiss Franc (CME) - 125,000 francs; $ per franc</td>
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<td>.5902</td>
<td>.5835</td>
<td>.5992</td>
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<td>.5540</td>
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<td>.5830</td>
<td>.5895</td>
<td>+ .0061</td>
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<td>.5813</td>
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<td>Est vol 5,676; vol Fri 6,330; open int 47,871, −597.</td>
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<tr>
<td>Australian Dollar (CME) - 100,000 drs.; $ per A.$</td>
<td>.5076</td>
<td>.5102</td>
<td>.5074</td>
<td>.5098</td>
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<td>.4810</td>
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<td>.5046</td>
<td>.5070</td>
<td>+ .0031</td>
<td>.5218</td>
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<tr>
<td>Mexican Peso (CME) - 500,000 new Mex. peso, $ per MP</td>
<td>.10843</td>
<td>.10850</td>
<td>.10810</td>
<td>.10835</td>
<td>+ .0006</td>
<td>.10940</td>
<td>.09770</td>
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<td>June</td>
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<tr>
<td>Est vol 1,940; vol Fri 2,817; open int 30,163, +626.</td>
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<tr>
<td>Euro FX (CME) - Euro 125,000; $ per Euro</td>
<td>.8681</td>
<td>.8694</td>
<td>.8593</td>
<td>.8686</td>
<td>+ .0085</td>
<td>.9630</td>
<td>.8336</td>
<td>103,910</td>
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<td>.8663</td>
<td>.8564</td>
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<td>+ .0085</td>
<td>.9275</td>
<td>.8365</td>
<td>1,492</td>
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<tr>
<td>Dec</td>
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<td></td>
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<tr>
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</tr>
</tbody>
</table>

- Widely used to hedge against changes in exchange rates
- WSJ listing:
Currency contracts: pricing

- Currency prepaid forward
  - Suppose you want to purchase ¥1 one year from today using $s
  - $P_{0,T} = x_0 e^{-r_y T}$
    - $x_0$ is current ($/$ ¥) exchange rate, and $r_y$ is the yen-denominated interest rate
    - Why? By deferring delivery of the currency one loses interest income from bonds denominated in that currency

- Currency forward
  - $F_{0,T} = x_0 e^{(r-r_y)T}$
    - $r$ is the $/$-denominated domestic interest rate
    - $F_{0,T} > x_0$ if $r > r_y$ (domestic risk-free rate exceeds foreign risk-free rate)
Currency contracts: pricing (cont.)

• Example 5.3:
  - ¥-denominated interest rate is 2% and current ($/ ¥) exchange rate is 0.009. To have ¥1 in one year one needs to invest today:
    - $0.009/¥ \times ¥1 \times e^{-0.02} = $0.008822

• Example 5.4:
  - ¥-denominated interest rate is 2% and $-denominated rate is 6%. The current ($/ ¥) exchange rate is 0.009. The 1-year forward rate:
    - $0.009e^{0.06-0.02} = 0.009367
Currency contracts: pricing (cont.)

- Synthetic currency forward: borrowing in one currency and lending in another creates the same cash flow as a forward contract.
- Covered interest arbitrage: offset the synthetic forward position with an actual forward contract.

### Table 5.12

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Year 0</th>
<th>Year 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td>¥</td>
</tr>
<tr>
<td>Borrow $xe^{-r_y}$ Dollar at 6% ($)</td>
<td>+0.008822</td>
<td>—</td>
</tr>
<tr>
<td>Convert to Yen @ 0.009 $/¥</td>
<td>—0.008822</td>
<td>+0.9802</td>
</tr>
<tr>
<td>Invest in Yen-Denominated Bill (¥)</td>
<td>—</td>
<td>—0.9802</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>0</td>
</tr>
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</table>
Eurodollar futures

- WSJ listing

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<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHANGE</th>
<th>YIELD</th>
<th>CHANGE</th>
<th>OPEN</th>
<th>INT.</th>
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<tbody>
<tr>
<td>Treasury Bills (CME) $1 mil.; pts of 100%</td>
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<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>Mar</td>
<td></td>
<td></td>
<td></td>
<td>98.26</td>
<td>+ .01</td>
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</table>

- Contract specifications

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<th>SETTLE</th>
<th>CHANGE</th>
<th>YIELD</th>
<th>CHANGE</th>
<th>OPEN</th>
<th>INT.</th>
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</thead>
<tbody>
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<td>Eurodollar (CME) $1 Million; pts of 100%</td>
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<tr>
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<td>98.09</td>
<td>98.08</td>
<td>98.09</td>
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<td>97.96</td>
<td>97.94</td>
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</tbody>
</table>

Specifications for the Eurodollar futures contract.

Where traded  Chicaco Mercantile Exchange

Size  3-month Eurodollar time deposit, $1 million principal

Months  Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month

Trading ends  5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.

Delivery  Cash settlement

Settlement  100 — British Banker’s Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)
Introduction to Commodity Forwards

- Commodity forward prices can be described by the same formula as that for financial forward prices:

\[ F_{0,T} = S_0 e^{(r-\delta)T} \]

- For financial assets, \( \delta \) is the dividend yield. For commodities, \( \delta \) is the commodity lease rate. The lease rate is the return that makes an investor willing to buy and then lend a commodity.

- The lease rate for a commodity can typically be estimated only by observing the forward prices.
Introduction to Commodity Forwards

- The set of prices for different expiration dates for a given commodity is called the forward curve (or the forward strip) for that date.

- If on a given date the forward curve is upward-sloping, then the market is in contango. If the forward curve is downward sloping, the market is in backwardation.

Note that forward curves can have portions in backwardation and portions in contango.