Lecture 04: Risk Preferences and Expected Utility Theory

• Prof. Markus K. Brunnermeier
Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance
3. vNM expected utility theory
   a) Intuition
   b) Axiomatic foundations
4. Risk aversion coefficients and portfolio choice
5. Prudence coefficient and precautionary savings
6. Mean-variance preferences
State-by-state Dominance

- State-by-state dominance ⬤ incomplete ranking
- « riskier »

Table 2.1 Asset Payoffs ($)

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cost at t=0</td>
<td>Value at t=1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>π₁ = π₂ = ½</td>
</tr>
<tr>
<td></td>
<td>s = 1</td>
<td>s = 2</td>
</tr>
<tr>
<td>investment 1</td>
<td>- 1000</td>
<td>1050</td>
</tr>
<tr>
<td>investment 2</td>
<td>- 1000</td>
<td>500</td>
</tr>
<tr>
<td>investment 3</td>
<td>- 1000</td>
<td>1050</td>
</tr>
</tbody>
</table>

- investment 3 state by state dominates 1.
State-by-state Dominance (ctd.)

Table 2.2 State Contingent ROR (r)

<table>
<thead>
<tr>
<th>State Contingent ROR (r)</th>
<th>s = 1</th>
<th>s = 2</th>
<th>Er</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment 1</td>
<td>5%</td>
<td>20%</td>
<td>12.5%</td>
<td>7.5%</td>
</tr>
<tr>
<td>Investment 2</td>
<td>-50%</td>
<td>60%</td>
<td>5%</td>
<td>55%</td>
</tr>
<tr>
<td>Investment 3</td>
<td>5%</td>
<td>60%</td>
<td>32.5%</td>
<td>27.5%</td>
</tr>
</tbody>
</table>

- Investment 1 mean-variance dominates 2
- BUT investment 3 does not m-v dominate 1!
State-by-state Dominance (ctd.)

Table 2.3 State Contingent Rates of Return

<table>
<thead>
<tr>
<th></th>
<th>State Contingent Rates of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s = 1</td>
</tr>
<tr>
<td>investment 4</td>
<td>3%</td>
</tr>
<tr>
<td>investment 5</td>
<td>3%</td>
</tr>
</tbody>
</table>

\[ \pi_1 = \pi_2 = \frac{1}{2} \]

\[ \begin{align*}
E[r_4] &= 4\%; \quad \sigma_4 = 1\% \\
E[r_5] &= 5.5\%; \quad \sigma_5 = 2.5\%
\end{align*} \]

- What is the trade-off between risk and expected return?
- Investment 4 has a higher Sharpe ratio \((E[r] - r^f) / \sigma\) than investment 5 for \(r^f = 0\).
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   c) Risk aversion coefficients
4. Risk aversion coefficients and portfolio choice
5. Prudence coefficient and precautionary savings
6. Mean-variance preferences

[DD4]
[L4]
[DD3]
[DD4, L4]
[DD5, L4]
[DD5]
[L4.6]
Stochastic Dominance

- Stochastic dominance can be defined independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. ("risk-preference-free")
- Less “demanding” than state-by-state dominance
Stochastic Dominance

- Still incomplete ordering
  - “More complete” than state-by-state ordering
  - State-by-state dominance $\Rightarrow$ stochastic dominance
  - Risk preference not needed for ranking!
    - independently of the specific trade-offs (between return, risk and other characteristics of probability distributions) represented by an agent's utility function. (“risk-preference-free”)

- Next Section:
  - Complete preference ordering and utility representations

*Homework:* Provide an example which can be ranked according to FSD, but not according to state dominance.
### Table 3-1 Sample Investment Alternatives

<table>
<thead>
<tr>
<th>States of nature</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoffs</td>
<td>10</td>
<td>100</td>
<td>2000</td>
</tr>
<tr>
<td>Proba $Z_1$</td>
<td>.4</td>
<td>.6</td>
<td>0</td>
</tr>
<tr>
<td>Proba $Z_2$</td>
<td>.4</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

$EZ_1 = 64, \sigma_{z_1} = 44$

$EZ_2 = 444, \sigma_{z_2} = 779$
First Order Stochastic Dominance

- **Definition 3.1**: Let $F_A(x)$ and $F_B(x)$, respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval $[a,b]$. We say that $F_A(x)$ **first order stochastically dominates (FSD)** $F_B(x)$ if and only if for all $x \in [a,b]$

  $$F_A(x) \leq F_B(x)$$

*Homework*: Provide an example which can be ranked according to FSD, but not according to state dominance.
First Order Stochastic Dominance

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xlabel=$x$,
    ylabel,]
    \addplot coordinates {
        (0,0) (1,0.1) (2,0.2) (3,0.3) (4,0.4) (5,0.5) (6,0.6) (7,0.7) (8,0.8) (9,0.9) (10,1)
    } node[yshift=0.5cm] {FA};
    \addplot coordinates {
        (0,0) (1,0.1) (2,0.2) (3,0.3) (4,0.4) (5,0.5) (6,0.6) (7,0.7) (8,0.8) (9,0.9) (10,1)
    } node[yshift=0.5cm] {FB};
\end{axis}
\end{tikzpicture}
\end{center}
Table 3-2 Two Independent Investments

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Prob.</th>
<th>Payoff</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.25</td>
<td>1</td>
<td>0.33</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>6</td>
<td>0.33</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
<td>8</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Figure 3-6 Second Order Stochastic Dominance Illustrated
Second Order Stochastic Dominance

Definition 3.2: Let \( F_A(\bar{x}) \), \( F_B(\bar{x}) \), be two cumulative probability distribution for random payoffs in \([a, b]\). We say that \( F_A(\bar{x}) \) second order stochastically dominates (SSD) \( F_B(\bar{x}) \) if and only if for any \( x \) :

\[
\int_{-\infty}^{\bar{x}} \left[ F_B(t) - F_A(t) \right] \, dt \geq 0
\]

(with strict inequality for some meaningful interval of values of \( t \)).
Mean Preserving Spread

\[ x_B = x_A + z \]  \hspace{1cm} (3.8)

where \( z \) is independent of \( x_A \) and has zero mean

for normal distributions

\[ \mu = \int x f_A(x) dx = \int x f_B(x) dx \]

\( \tilde{x} \), Payoff

Figure 3-7  Mean Preserving Spread
Mean Preserving Spread & SSD

Theorem 3.4: Let \( F_A(\cdot) \) and \( F_B(\cdot) \) be two distribution functions defined on the same state space with identical means. Then the follow statements are equivalent:

- \( F_A(\tilde{x}) \) SSD \( F_B(\tilde{x}) \)
- \( F_B(\tilde{x}) \) is a mean preserving spread of \( F_A(\tilde{x}) \) in the sense of Equation (3.8) above.
Overview: Risk Preferences

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5. Prudence coefficient and precautionary savings [DD5]
6. Mean-variance preferences [L4.6]
A Hypothetical Gamble

- Suppose someone offers you this gamble:
  - "I have a fair coin here. I'll flip it, and if it's tail I pay you $1 and the gamble is over. If it's head, I'll flip again. If it's tail then, I pay you $2, if not I'll flip again. With every round, I double the amount I will pay to you if it's tail."

- Sounds like a good deal. After all, you can't loose. So here's the question:
  - How much are you willing to pay to take this gamble?
Infinite Expected Value

- With probability 1/2 you get $1. \quad \left(\frac{1}{2}\right)^1 \text{ times } 2^0
- With probability 1/4 you get $2. \quad \left(\frac{1}{2}\right)^2 \text{ times } 2^1
- With probability 1/8 you get $4. \quad \left(\frac{1}{2}\right)^3 \text{ times } 2^2
- etc.

The expected payoff is the sum of these payoffs, weighted with their probabilities, so

$$\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \cdot 2^{t-1} = \sum_{t=1}^{\infty} \frac{1}{2} = \infty$$
An Infinitely Valuable Gamble?

- I should pay everything I own \textit{and more} to purchase the right to take this gamble!
- Yet, in practice, no one is prepared to pay such a high price. Why?
- Even though the expected payoff is infinite, the distribution of payoffs is not attractive…
- With 93\% probability we get $8 or less, with 99\% probability we get $64 or less.
What Should We Do?

- How can we decide in a rational fashion about such gambles (or investments)?
- Bernoulli suggests that large gains should be weighted less. He suggests to use the natural logarithm. [Cremer - another great mathematician of the time - suggests the square root.]

\[
\sum_{t=1}^{\infty} \left( \frac{1}{2} \right)^t \cdot \ln(2^{t-1}) = \ln(2) = \text{expected utility of gamble} < \infty
\]

\[
\text{probability of payoff utility of payoff}
\]

Bernoulli would have paid at most \(e^{\ln(2)} = \$2\) to participate in this gamble.
Risk-Aversion and Concavity

- Von Neumann and Morgenstern prove that with these assumptions, one can represent a utility over lotteries as an expected utility.

- The shape of the von Neumann Morgenstern (NM) utility function contains a lot of information.

Consider a fifty-fifty lottery with two prizes...
Risk-aversion and concavity

- Risk-aversion means that the certainty equivalent is smaller than the expected prize.
  - We conclude that a risk-averse NM utility function must be concave.
Jensen’s Inequality

Theorem 3.1 (Jensen’s Inequality):

- Let $g(\cdot)$ be a concave function on the interval $[a,b]$, and $x$ be a random variable such that $\text{Prob}(x \in [a,b]) = 1$
- Suppose the expectations $E(x)$ and $E[g(x)]$ exist; then
  $$E[g(\bar{x})] \leq g[E(\bar{x})]$$

Furthermore, if $g(\cdot)$ is strictly concave, then the inequality is strict.
Representation of Preferences

A preference ordering is (i) complete, (ii) transitive, (iii) continuous and (iv) relatively stable can be represented by a utility function, i.e.

\[(c_0, c_1, \ldots, c_S) \succ (c'_0, c'_1, \ldots, c'_S) \iff U(c_0, c_1, \ldots, c_S) > U(c'_0, c'_1, \ldots, c'_S)\]

(preference ordering over lotteries – (S+1)-dimensional space)
Preferences over Prob. Distributions

- Consider $c_0$ fixed, $c_1$ is a random variable
- Preference ordering over probability distributions
- Let
  - $P$ be a set of probability distributions with a finite support over a set $X$,
  - $\succ$ a (strict) preference ordering over $P$, and
  - Define $\succsim$ by $p \succsim q$ if $q \not\succ p$
- $S$ states of the world
- Set of all possible lotteries

$$P = \{ p \in \mathbb{R}^S | p(c) \geq 0, \sum p(c) = 1 \}$$

- Space with $S$ dimensions

- Can we simplify the utility representation of preferences over lotteries?
- Space with *one* dimension – income
- We need to assume further axioms
Expected Utility Theory

- A binary relation that satisfies the following three axioms if and only if there exists a function $u(\cdot)$ such that

$$p \succ q \iff \sum u(c) \ p(c) > \sum u(c) \ q(c)$$

i.e. preferences correspond to expected utility.
vNM Expected Utility Theory

- **Axiom 1 (Completeness and Transitivity):**
  - Agents have preference relation over P (repeated)

- **Axiom 2 (Substitution/Independence)**
  - For all lotteries $p, q, r \in P$ and $\alpha \in (0,1]$, $p \succ q$ iff $\alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r$ (see next slide)

- **Axiom 3 (Archimedian/Continuity)**
  - For all lotteries $p, q, r \in P$, if $p \succ q \succ r$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha) r \succ q \succ \beta p + (1 - \beta) r$.

*Problem:* $p$ you get $100 for sure, $q$ you get $10 for sure, $r$ you are killed
Independence Axiom

- Independence of irrelevant alternatives:

\[ p \succeq q \iff \pi \ simeq \ p \ \pi \ \ simeq \ \ r \ \ r \]
Allais Paradox –
Violation of Independence Axiom
Allais Paradox –
Violation of Independence Axiom

\[ \begin{align*}
10\% & \rightarrow 10' & \succ
0 & \rightarrow 0 \\
100\% & \rightarrow 10' & \succ
0 & \rightarrow 0 \\
9\% & \rightarrow 15' & \succ
0 & \rightarrow 0 \\
90\% & \rightarrow 15' & \succ
0 & \rightarrow 0
\end{align*} \]
Allais Paradox – Violation of Independence Axiom
vNM EU Theorem

- A binary relation that satisfies the axioms 1-3 if and only if there exists a function $u(\bullet)$ such that

$$p \succ q \iff \sum u(c) \cdot p(c) > \sum u(c) \cdot q(c)$$

i.e. preferences correspond to expected utility.
Expected Utility Theory

\[ U(Y) \]

- \( U(Y_0 + Z_2) \)
- \( U(Y_0 + E(\tilde{Z})) \)
- \( EU(Y_0 + \tilde{Z}) \)
- \( U(Y_0 + Z_1) \)

Y_0, Y_0 + Z_1, CE(Y_0 + \tilde{Z}), Y_0 + E(\tilde{Z}), Y_0 + Z_2
Expected Utility & Stochastic Dominance

- **Theorem 3.2**: Let \( F_A(\tilde{x}) \), \( F_B(\tilde{x}) \), be two cumulative probability distribution for random payoffs \( \tilde{x} \in [a, b] \). Then \( F_A(\tilde{x}) \) FSD \( F_B(\tilde{x}) \) if and only if for all non decreasing utility functions \( U(\cdot) \).

\[
E_A U(\tilde{x}) \geq E_B U(\tilde{x})
\]
Expected Utility & Stochastic Dominance

- **Theorem 3.3:** Let $F_A(x)$, $F_B(x)$, be two cumulative probability distribution for random payoffs $x$ defined on $[a, b]$. Then, $F_A(x)$ SSD $F_B(x)$ if and only if $E_A U(x) \geq E_B U(x)$ for all non decreasing and concave $U$. 
Digression: Subjective EU Theory

- Derive perceived probability from preferences!
  - Set S of prizes/consequences
  - Set Z of states
  - Set of functions $f(s) \in Z$, called acts (consumption plans)
- Seven SAVAGE Axioms
  - Goes beyond scope of this course.
Digression: Ellsberg Paradox

- 10 balls in an urn
  Lottery 1: win $100 if you draw a red ball
  Lottery 2: win $100 if you draw a blue ball
- Uncertainty: Probability distribution is not known
- Risk: Probability distribution is known
  (5 balls are red, 5 balls are blue)

- Individuals are “uncertainty/ambiguity averse”
  (non-additive probability approach)
Digression: Prospect Theory

- Value function (over gains and losses)

- Overweight low probability events
- Experimental evidence
Indifference curves

Any point in this plane is a particular lottery.

Where is the set of risk-free lotteries?

If $x_1 = x_2$, then the lottery contains no risk.
Indifference curves

Where is the set of lotteries with expected prize $E[L]=z$?

It's a straight line, and the slope is given by the relative probabilities of the two states.
Suppose the agent is risk averse. Where is the set of lotteries which are indifferent to $(z, z)$?

That's not right! Note that there are risky lotteries with smaller expected prize and which are preferred.
Indifference curves

So the indifference curve must be tangent to the iso-expected-prize line.

This is a direct implication of risk-aversion alone.
Indifference curves

But risk-aversion does not imply convexity.

This indifference curve is also compatible with risk-aversion.
Indifference curves

The tangency implies that the gradient of \( \mathbf{v} \) at the point \((z,z)\) is collinear to \( \pi \).

Formally, \( \nabla \mathbf{v}(z,z) = \lambda \pi \), for some \( \lambda > 0 \).
Certainty Equivalent and Risk Premium

(3.6) \[ \text{EU}(Y + \tilde{Z}) = U(Y + CE(Y, \tilde{Z})) \]

(3.7) \[ = U(Y + E\tilde{Z} - \Pi(Y, \tilde{Z})) \]
Certainty Equivalent and Risk Premium

Figure 3-3 Certainty Equivalent and Risk Premium: An Illustration
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5. Prudence coefficient and precautionary savings [DD5]
6. Mean-variance preferences [L4.6]
Measuring Risk aversion

Figure 3-1 A Strictly Concave Utility Function
Arrow-Pratt measures of risk aversion and their interpretations

- **absolute risk aversion** \( = - \frac{U''(Y)}{U'(Y)} \equiv R_A(Y) \)

- **relative risk aversion** \( = - \frac{Y U''(Y)}{U'(Y)} \equiv R_R(Y) \)

- **risk tolerance** \( = \frac{1}{R_A} \)
Absolute risk aversion coefficient

\[ R_A = -\frac{U''(Y)}{U'(Y)} \]

\[ \pi(Y, h) = \frac{1}{2} + \frac{1}{4}hR_A(Y) + \text{HOT} \]
Relative risk aversion coefficient

\[ R_R = -\frac{U''(Y)}{U'(Y)} Y \]

\[ \pi(Y, \theta) = \frac{1}{2} + \frac{1}{4} \theta R_R(Y) + \text{HOT} \]

*Homework:* Derive this result.
CARA and CRRA-utility functions

- Constant Absolute RA utility function
  \[ U(Y) = -e^{-\rho Y} \]

- Constant Relative RA utility function
  \[ U(Y) = \begin{cases} \frac{Y^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \ln Y & \text{for } \gamma = 1 \end{cases} \]
**Investor’s Level of Relative Risk Aversion**

\[ \frac{(Y + CE)^{1-\gamma}}{1-\gamma} = \frac{1}{2} \frac{(Y + 50,000)^{1-\gamma}}{1-\gamma} + \frac{1}{2} \frac{(Y + 100,000)^{1-\gamma}}{1-\gamma} \]

- \(Y=0\)
  - \(\gamma = 0\) \(CE = 75,000\) (risk neutrality)
  - \(\gamma = 1\) \(CE = 70,711\)
  - \(\gamma = 2\) \(CE = 66,246\)
  - \(\gamma = 5\) \(CE = 58,566\)
  - \(\gamma = 10\) \(CE = 53,991\)
  - \(\gamma = 20\) \(CE = 51,858\)
  - \(\gamma = 30\) \(CE = 51,209\)

- \(Y=100,000\)
  - \(\gamma = 5\) \(CE = 66,530\)
Risk aversion and Portfolio Allocation

- No savings decision (consumption occurs only at $t=1$)
- Asset structure
  - One risk free bond with net return $r_f$
  - One risky asset with random net return $r$ ($a =$quantity of risky assets)

\[
\max_a E\left[U(Y_0(1 + r_f) + a(r - r_f))\right]
\]

FOC:

\[
E[U'(Y_0(1 + r_f) + a(r - r_f))(r - r_f)] = 0
\]
Theorem 4.1: Assume \( U'(\cdot) > 0 \), and \( U''(\cdot) < 0 \) and let \( \hat{\alpha} \) denote the solution to above problem. Then

\[
\hat{\alpha} > 0 \text{ if and only if } E\tilde{r} > r_f
\]
\[
\hat{\alpha} = 0 \text{ if and only if } E\tilde{r} = r_f
\]
\[
\hat{\alpha} < 0 \text{ if and only if } E\tilde{r} < r_f.
\]

Define \( W(a) = E\{U(Y_0(1 + r_f) + a(\tilde{r} - r_f))\} \). The FOC can then be written

\[
W'(a) = E[U'(Y_0(1 + r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)] = 0.
\]

By risk aversion (\( U'' < 0 \)), \( W''(a) = E[U''(Y_0(1 + r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)^2] < 0 \), that is, \( W'(a) \) is everywhere decreasing. It follows that \( \hat{\alpha} \) will be positive if and only if \( W'(0) = U'(Y_0(1 + r_f))E(\tilde{r} - r_f) > 0 \) (since then \( a \) will have to be increased from its value of 0 to achieve equality in the FOC). Since \( U' \) is always strictly positive, this implies \( \hat{\alpha} > 0 \) if and only if \( E(\tilde{r} - r_f) > 0 \).

The other assertion follows similarly. \( \square \)
Portfolio as wealth changes

- Theorem 4.4 (Arrow, 1971): Let \( \hat{a} = \hat{a}(Y_0) \) be the solution to max-problem above; then:

  (i) \( \frac{\partial R_A}{\partial Y} < 0 \) (DARA) implies \( \frac{\partial \hat{a}}{\partial Y_0} > 0 \)

  (ii) \( \frac{\partial R_A}{\partial Y} = 0 \) (CARA) implies \( \frac{\partial \hat{a}}{\partial Y_0} = 0 \)

  (iii) \( \frac{\partial R_A}{\partial Y} > 0 \) (IARA) implies \( \frac{\partial \hat{a}}{\partial Y_0} < 0 \)
Portfolio as wealth changes

• **Theorem 4.5 (Arrow 1971):** If, for all wealth levels $Y$,

  \[
  \begin{align*}
  &\text{(i) } \frac{\partial R_R}{\partial Y} = 0 \text{ (CRRA) implies } \eta = 1 \\
  &\text{(ii) } \frac{\partial R_R}{\partial Y} < 0 \text{ (DRRA) implies } \eta > 1 \\
  &\text{(iii) } \frac{\partial R_R}{\partial Y} > 0 \text{ (IRRA) implies } \eta < 1 \\
  \end{align*}
  \]

where $\eta = \frac{da/a}{dY/Y}$ (elasticity)
Log utility & Portfolio Allocation

\[ U(Y) = \ln Y. \]

\[ E \left\{ \frac{\tilde{r} - r_f}{Y_0(1+r_f) + a(\tilde{r} - r_f)} \right\} = 0 \]

2 states, where \( r_2 > r_f > r_1 \)

\[ \frac{a}{Y_0} = \frac{(1+r_f)[E[\tilde{r}] - r_f]}{-(r_1-r_f)(r_2-r_f)} > 0 \]

Constant fraction of wealth is invested in risky asset!
Portfolio of risky assets as wealth changes

Now -- many risky assets

- **Theorem 4.6 (Cass and Stiglitz, 1970).** Let the vector

\[
\begin{bmatrix}
\hat{a}_1(Y_0) \\
\vdots \\
\hat{a}_j(Y_0)
\end{bmatrix}
\]

\[\vdots \]

denote the amount optimally invested in the \( J \) risky assets if

the wealth level is \( Y_0 \). Then

\[
\begin{bmatrix}
\hat{a}_1(Y_0) \\
\vdots \\
\hat{a}_j(Y_0)
\end{bmatrix} = \begin{bmatrix}
a_1 \\
\vdots \\
a_j
\end{bmatrix} f(Y_0)
\]

if and only if either

(i) \[ U'(Y_0) = (\theta Y_0 + \kappa)^\lambda \]

or

(ii) \[ U'(Y_0) = \xi e^{-\nu Y_0} . \]

- In words, it is sufficient to offer a **mutual fund**.
LRT/HARA-utility functions

- Linear Risk Tolerance/hyperbolic absolute risk aversion

\[- \frac{u''(c)}{u'(c)} = \frac{1}{A + Bc}\]

- Special Cases
  - B=0, A>0   CARA
  - B ≠ 0, ≠1 Generalized Power
    - B=1   Log utility
    - B=-1  Quadratic Utility
    - B ≠ 1 A=0   CRRA Utility function

\[u(c) = \frac{1}{B-1}(A + Bc)^{\frac{B-1}{B}}\]

\[u(c) = \ln (A+Bc)\]

\[u(c)=-(A-c)^2\]

\[u(c) = \frac{1}{B-1}(Bc)^{\frac{B-1}{B}}\]
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5. Prudence coefficient and precautionary savings
6. Mean-variance preferences
Introducing Savings

• Introduce savings decision: Consumption at $t=0$ and $t=1$

• *Asset structure 1*:
  - risk free bond $R^f$
  - NO risky asset with random return
  - Increase $R^f$:
    - **Substitution effect**: shift consumption from $t=0$ to $t=1$
      $\Rightarrow$ save more
    - **Income effect**: agent is “effectively richer” and wants to consume some of the additional richness at $t=0$
      $\Rightarrow$ save less
    - For log-utility ($\gamma=1$) both effects cancel each other
Prudence and Pre-cautionary Savings

• Introduce savings decision Consumption at $t=0$ and $t=1$

• Asset structure 2:
  – NO risk free bond
  – One risky asset with random gross return $R$
Prudence and Savings Behavior

- Risk aversion is about the willingness to insure …
- … but not about its comparative statics.
- How does the behavior of an agent change when we marginally increase his exposure to risk?
- An old hypothesis (going back at least to J.M. Keynes) is that people should save more now when they face greater uncertainty in the future.
- The idea is called precautionary saving and has intuitive appeal.
Prudence and Pre-cautionary Savings

- Does not directly follow from risk aversion alone.
- Involves the third derivative of the utility function.
- Kimball (1990) defines **absolute prudence** as 
  \[ P(w) := -u'''(w)/u''(w). \]
- Precautionary saving if and only if they are prudent.
- This finding is important when one does comparative statics of interest rates.
- Prudence seems uncontroversial, because it is weaker than DARA.
Pre-cautionary Saving (extra material)

\[
\max_s E[U(Y_0 - s) + \delta U(sR)]
\]

s.t. \( s \geq 0 \)

FOC: \( U'(Y_0 - s) = \delta E[U'(sR)R] \)

(+) in \( s \)

(-) in \( s \)

Is saving \( s \) increasing/decreasing in risk of \( R \)?
Is RHS increasing/decreasing is riskiness of \( R \)?
Is \( U'() \) convex/concave?
Depends on third derivative of \( U() \! \)

N.B: For \( U(c) = \ln c \), \( U'(sR)R = 1/s \) does not depend on \( R \).
Pre-cautionary Saving (extra material)

2 effects: Tomorrow consumption is more volatile
- consume more today, since it’s not risky
- save more for precautionary reasons

Theorem 4.7 (Rothschild and Stiglitz, 1971): Let \( \tilde{R}_A \) and \( \tilde{R}_B \) be two return distributions with identical means such that \( \tilde{R}_B = \tilde{R}_A + e \), (where \( e \) is white noise) and let \( s_A \) and \( s_B \) be, respectively, the savings out of \( Y_0 \) corresponding to the return distributions \( \tilde{R}_A \) and \( \tilde{R}_B \).

If \( R'_R(Y) \leq 0 \) and \( R_R(Y) > 1 \), then \( s_A < s_B \);
If \( R'_R(Y) \geq 0 \) and \( R_R(Y) < 1 \), then \( s_A > s_B \)
Prudence & Pre-cautionary Saving

\[ P(c) = \frac{-U'''(c)}{U''(c)} \]

\[ P(c)c = \frac{-cU'''(c)}{U''(c)} \]

- **Theorem 4.8**: Let \( \tilde{R}_A, \tilde{R}_B \) be two return distributions such that \( \tilde{R}_A \sim SSD \tilde{R}_B \), and let \( s_A \) and \( s_B \) be, respectively, the savings out of \( Y_0 \) corresponding to the return distributions \( \tilde{R}_A \) and \( \tilde{R}_B \). Then,

  \[ s_A \geq s_B \quad \text{iff} \quad cP(c) \leq 2, \quad \text{and conversely}, \]

  \[ s_A < s_B \quad \text{iff} \quad cP(c) > 2 \]
Overview: Risk Preferences

1. State-by-state dominance
2. Stochastic dominance
3. vNM expected utility theory
   a) Intuition
   b) Axiomatic foundations
4. Risk aversion coefficients and portfolio choice
5. Prudence coefficient and precautionary savings
6. Mean-variance preferences
Mean-variance Preferences

- Early researchers in finance, such as Markowitz and Sharpe, used just the mean and the variance of the return rate of an asset to describe it.
- Mean-variance characterization is often easier than using an vNM utility function.
- But is it compatible with vNM theory?
- The answer is yes … approximately … under some conditions.
Mean-Variance: quadratic utility

Suppose utility is quadratic, \( u(y) = ay - by^2 \).

Expected utility is then
\[
E[u(y)] = aE[y] - bE[y^2]
\]
\[
= aE[y] - b(E[y]^2 + \text{var}[y]).
\]

Thus, expected utility is a function of the mean, \( E[y] \), and the variance, \( \text{var}[y] \), only.
Mean-Variance: joint normals

- Suppose all lotteries in the domain have normally distributed prized. (independence is not needed).
  - This requires an infinite state space.
- Any linear combination of normals is also normal.
- The normal distribution is completely described by its first two moments.
- Hence, expected utility can be expressed as a function of just these two numbers as well.
Mean-Variance: linear distribution classes

- Generalization of joint normals.
- Consider a class of distributions $F_1, \ldots, F_n$ with the following property:
  - for all $i$ there exists $(m,s)$ such that $F_i(x) = F_1(a+bx)$ for all $x$.
- This is called a linear distribution class.
- It means that any $F_i$ can be transformed into an $F_j$ by an appropriate shift $(a)$ and stretch $(b)$.
- Let $y_i$ be a random variable drawn from $F_i$. Let $\mu_i = \text{E}\{y_i\}$ and $\sigma_i^2 = \text{E}\{(y_i-\mu_i)^2\}$ denote the mean and the var of $y_i$. 

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Mean-Variance: linear distribution classes

- Define then the random variable $x = (y_i - \mu_i)/\sigma_i$. We denote the distribution of $x$ with $F$.
- Note that the mean of $x$ is 0 and the variance is 1, and $F$ is part of the same linear distribution class.
- Moreover, the distribution of $x$ is independent of which $i$ we start with.
  - We want to evaluate the expected utility of $y_i$, 
  \[ \int_{-\infty}^{+\infty} v(z) dF_i(z). \]
Mean-Variance: linear distribution classes

But $y_i = \mu_i + \sigma_i x$, thus

$$\int_{-\infty}^{+\infty} \nu(z) dF_i(z) = \int_{-\infty}^{+\infty} \nu(\mu_i + \sigma_i z) dF(z)$$

$$=: u(\mu_i, \sigma_i).$$

The expected utility of all random variables drawn from the same linear distribution class can be expressed as functions of the mean and the standard deviation only.
Mean-Variance: small risks

- Justification for mean-variance for the case of small risks.
- Use a second order (local) Taylor approximation of vNM U(c).
- If U(c) is concave, second order Taylor approximation is a quadratic function with a negative coefficient on the quadratic term.
- Expectation of a quadratic utility function can be evaluated with the mean and variance.
Mean-Variance: small risks

- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function. The Taylor approximation is

$$f(x) \approx f(x_0) + f'(x_0) \frac{(x - x_0)^1}{1!} + f''(x_0) \frac{(x - x_0)^2}{2!} + f'''(x_0) \frac{(x - x_0)^3}{3!} + \cdots$$

- Use Taylor approximation for $E[u(x)]$. 
Mean-Variance: small risks

- Since $E[u(w+x)] = u(c^{CE})$, this simplifies to

$$w - c^{CE} \approx R_A(w) \frac{\text{var}(x)}{2}.$$ 

- $w - c^{CE}$ is the risk premium.

- We see here that the risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk.
Mean-Variance: small risks

- The same exercise can be done with a multiplicative risk.
- Let $y = gw$, where $g$ is a positive random variable with unit mean.
- Doing the same steps as before leads to

$$1 - \kappa \approx R_R(w) \frac{\text{var}(g)}{2},$$

where $\kappa$ is the certainty equivalent growth rate, $u(\kappa w) = E[u(gw)]$.

- The coefficient of relative risk aversion is relevant for multiplicative risk, absolute risk aversion for additive risk.
Extra material follows!
Joint saving-portfolio problem

• Consumption at $t=0$ and $t=1$. (savings decision)

• Asset structure
  – One risk free bond with net return $r_f$
  – One risky asset ($a =$ quantity of risky assets)

$$\max_{\{a, s\}} U(Y_0 - s) + \delta EU(s(1 + r_f) + a(\tilde{r} - r_f))$$  \hspace{1cm} (4.7)

FOC:

s: $U'(c_t) = \delta E[U'(c_{t+1})(1+r_f)]$

a: $E[U'(c_{t+1})(r-r_f)] = 0$
for CRRA utility functions

\[
\begin{align*}
    s: & \quad (Y_0 - s)^{-\gamma}(-1) + \delta E \left[ s(1 + r_f) + a(\tilde{r} - r_f) \right]^{-\gamma} (1 + r_f) = 0 \\
    a: & \quad E \left[ (s(1 + r_f) + a(\tilde{r} - r_f))^{-\gamma} (\tilde{r} - r_f) \right] = 0
\end{align*}
\]

Where \( s \) is total saving and \( a \) is amount invested in risky asset.
Multi-period Setting

- Canonical framework (exponential discounting)
  \[ U(c) = \mathbb{E}[ \sum \beta^t u(c_t)] \]
  - prefers earlier uncertainty resolution if it affect action
  - indifferent, if it does not affect action

- Time-inconsistent (hyperbolic discounting)
  Special case: \( \beta - \delta \) formulation
  \[ U(c) = \mathbb{E}[u(c_0) + \beta \sum \delta^t u(c_t)] \]
  Preference for the timing of uncertainty resolution
  recursive utility formulation (Kreps-Porteus 1978)
Multi-period Portfolio Choice

\[
\max \left\{ s_t, a_t \right\}_{t=0}^{T-1} E \left[ \sum_{t=0}^{T} \beta^t U(c_t) \right]
\]

s.t.
\[
c_T = s_{T-1}(1 + r_f) + a_{T-1}(r_T - r_f)
\]
\[
c_t + s_t \leq s_{t-1}(1 + r_f) + a_{t-1}(r_t - r_f)
\]
\[
c_0 + s_0 \leq Y_0
\]

Theorem 4.10 (Merton, 1971): Consider the above canonical multi-period consumption-saving-portfolio allocation problem. Suppose U() displays CRRA, \( r_f \) is constant and \{r\} is i.i.d. Then \( a/s_t \) is time invariant.
Digression: Preference for the timing of uncertainty resolution

Kreps-Porteus

\[ U_0(x_1, x_2(s)) = W(x_1, E[U_1(x_1, x_2(s))]) \]

Early (late) resolution if \( W(P_1, \ldots) \) is convex (concave)

Marginal rate of temporal substitution \( \equiv \) risk aversion