Lecture 06: Sharpe Ratio, Bounds and the Equity Premium Puzzle

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The long-term gains from the stock market have been astounding.

TODAY’S VALUE OF 1$ INVESTED IN 1972
Including reinvestment of interests and dividends

- Exponential growth of stock prices
- 1y-Bonds earned less, but grew very smoothly
- Stock account dipping below $1

Source: Mertens, Data from Ibbotson Associates
Sharpe Ratios and Bounds

• Consider a one period security available at date \( t \) with payoff \( x_{t+1} \). We have

\[
p_t = E_t[m_{t+1} x_{t+1}]
\]

or

\[
p_t = E_t[m_{t+1}] E_t[x_{t+1}] + \text{Cov}[m_{t+1}, x_{t+1}]
\]

• For a given \( m_{t+1} \) we let

\[
R^f_{t+1} = 1/ E_t[m_{t+1}]
\]

- Note that \( R^f_{t+1} \) will depend on the choice of \( m_{t+1} \) unless there exists a riskless portfolio
- \( R_{t+1} \) is the return from \( t \) to \( t+1 \) and typically measurable w.r.t. \( F_{t+1} \). (An exception is \( R^f \), which is measurable w.r.t. \( F_t \), but we stick with subscript \( t+1 \).)
Sharpe Ratios and Bounds (ctd.)

– Hence

\[ p_t = (1/R_{t+1}^f) E_t[x_{t+1}] + \text{Cov}[m_{t+1}, x_{t+1}] \]

– price = expected PV + Risk adjustment

– positive correlation with the discount factor adds value
in Returns

\[ E_t \left[ m_{t+1} \times_{t+1} \right] = p_t \]

- divide both sides by \( p_t \) and note that \( x_{t+1} = R_{t+1} \)

\[ E_t \left[ m_{t+1} R_{t+1} \right] = 1 \]  (vector)

- using \( R_{t+1}^f = \frac{1}{E_t[m_{t+1}]} \), we get

\[ E_t \left[ m_{t+1} (R_{t+1} - R_{t+1}^f) \right] = 0 \]

- \( m \)-discounted expected excess return for all assets is zero.
in Returns

- Since \( E_t [m_{t+1} (R_{t+1} - R_{t+1}^f)] = 0 \)

\[
\text{Cov}_t[m_{t+1}, R_{t+1} - R_{t+1}^f] = E_t[m_{t+1}(R_{t+1} - R_{t+1}^f)] - E_t[m_{t+1}]E_t[R_{t+1} - R_{t+1}^f]
\]

\[
= - E_t[m_{t+1}] E_t[R_{t+1} - R_{t+1}^f]
\]

- That is, risk premium or expected excess return

\[
E_t [R_{t+1} - R_{t+1}^f] = - \text{Cov}_t[m_{t+1}, R_{t+1}] / E[m_{t+1}]
\]

is determined by its covariance with the stochastic discount factor
Sharpe Ratio

• Multiply both sides with portfolio \( h \)

\[
E_t [(R_{t+1} - R_{t+1}^f)h] = - \frac{\text{Cov}_t[m_{t+1}, R_{t+1}h]}{E[m_{t+1}]} \]

• NB: All results also hold for unconditional expectations \( E[\cdot] \)

\[
E[(R_{t+1} - R_{t+1}^f)h] = - \frac{\rho(m_{t+1}, R_{t+1}h)\sigma(R_{t+1}h)\sigma(m_{t+1})}{E[m_{t+1}]} \]

• Rewritten in terms of Sharpe Ratio = ...

\[
- \frac{\sigma(m_{t+1})}{E[m_{t+1}]}[\rho(m_{t+1}, R_{t+1}h)] = \frac{E[(R_{t+1} - R_{t+1}^f)h]}{\sigma(R_{t+1}h)} \]
Hansen-Jagannathan Bound

- Since $\rho \in [-1,1]$ we have

$$\frac{\sigma(m_{t+1})}{E[m_{t+1}]} \geq \sup_h \left| \frac{E[(R_{t+1} - R_{t+1}^f)h]}{\sigma(R_{t+1}h)} \right|$$

- **Theorem (Hansen-Jagannathan Bound):**
  The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
Hansen-Jagannathan Bound

– Theorem (Hansen-Jagannathan Bound):
  The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
  – Can be used to easy check the “viability” of a proposed discount factor
  – Given a discount factor, this inequality bounds the available risk-return possibilities
  – The result also holds conditional on date $t$ info
Hansen-Jagannathan Bound

expected return

\[ R \]

\[ \sigma \]

slope \( \sigma (m) / E[m] \)

available portfolios
Assuming Expected Utility

- \( c_0 \in \mathbb{R}, \ c_1 \in \mathbb{R}^S \)
- \( U(c_0,c_1) = \sum_s \pi_s u(c_{0,s},c_{1,s}) \)
- \( \partial_0 u = \left( \frac{\partial u(c^*_0,c^*_1,1)}{\partial c_0}, \ldots, \frac{\partial u(c^*_0,c^*_1,S)}{\partial c_0} \right) \)
- \( \partial_1 u = \left( \frac{\partial u(c^*_0,c^*_1,1)}{\partial c_{1,1}}, \ldots, \frac{\partial u(c^*_0,c^*_1,S)}{\partial c_{1,S}} \right) \)
- Stochastic discount factor
  \[
m = \left( \frac{MRS}{\pi} \right) = \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \in \mathbb{IR}^S
\]
• **Digression:** if utility is in addition time-separable
  \[ u(c_0, c_1) = v(c_0) + v(c_1) \]

• Then
  \[
  \partial_0 u = \left( \frac{\partial v(c_0^*)}{\partial c_0}, \ldots, \frac{\partial v(c_0^*)}{\partial c_0} \right)
  \]
  \[
  \partial_1 u = \left( \frac{\partial v(c_1^*, 1)}{\partial c_1, 1}, \ldots, \frac{\partial v(c_1^*, S)}{\partial c_1, S} \right)
  \]

• and
  \[
  m_s = \frac{1}{\pi_s} \pi_s v'(c_1, s) = \frac{v'(c_1, s)}{v'(c_0)}
  \]
Equity Premium Puzzle

- Recall $E[R^j] - R^f = -R^f \text{Cov}[m, R^j]$
- Now: $E[R^j] - R^f = -R^f \text{Cov}[\partial_1 u, R^j]/E[\partial_0 u]$
- Recall Hansen-Jaganathan bound

\[
\sigma(m) \geq \left| \frac{E[(R-R^f)]}{\sigma(R)} \right|; \quad E[m] = \frac{1}{R^f}
\]

\[
\sigma(m) \geq \frac{1}{R^f} \left| \frac{E[(R-R^f)]}{\sigma(R)} \right|
\]
Equity Premium Puzzle (ctd.)

\[ \sigma \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \geq \frac{1}{R_f} \left| \frac{E[(R-R_f)]}{\sigma(R)} \right| \]

Equity Premium Puzzle:
- high observed Sharpe ratio of stock market indices
- low volatility of consumption
  \( \Rightarrow \) (unrealistically) high level of risk aversion
A simple example

- $S = 2$, $\pi_1 = \frac{1}{2}$,
- 3 securities with $x^1 = (1,0)$, $x^2 = (0,1)$, $x^3 = (1,1)$
- Let $m = (\frac{1}{2}, 1)$, $\sigma = \frac{1}{4} = \sqrt{\frac{1}{2}(\frac{1}{2} - \frac{3}{4})^2 + \frac{1}{2}(1 - \frac{3}{4})^2}$
- Hence, $p^1 = \frac{1}{4}$, $p^2 = \frac{1}{2}$, $p^3 = \frac{3}{4}$ and
- $R^1 = (4,0)$, $R^2 = (0,2)$, $R^3 = (\frac{4}{3}, \frac{4}{3})$
- $E[R^1] = 2$, $E[R^2] = 1$, $E[R^3] = \frac{4}{3}$
Example: Where does SDF come from?

- “representative agent” with
  - endowment: 1 in date 0, (2,1) in date 1
  - utility \( EU(c_0, c_1, c_2) = \sum_s \pi_s (\ln c_0 + \ln c_{1,s}) \)
  - i.e. \( u(c_0, c_{1,s}) = \ln c_0 + \ln c_{1,s} \) (additive) time separable u-function

\[ m = \frac{\partial_1 u(1,2,1)}{E[\partial_0 u(1,2,1)]} = (\frac{c_0}{c_{1,1}}, \frac{c_0}{c_{1,2}}) = (1/2, 1/1) \]

\[ m = (\frac{1}{2}, 1) \] since endowment=consumption

- Low consumption states are high “m-states”
- Risk-neutral probabilities combine true probabilities and marginal utilities.