Lecture 07: Mean-Variance Analysis & Capital Asset Pricing Model (CAPM)

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Overview

1. Simple CAPM with quadratic utility functions (derived from state-price beta model)

2. Mean-variance preferences
   – Portfolio Theory
   – CAPM (intuition)

3. CAPM
   – Projections
   – Pricing Kernel and Expectation Kernel
Recall State-price Beta model

Recall:
\[ E[R^h] - R^f = \beta^h E[R^* - R^f] \]

where \( \beta^h := \frac{\text{Cov}[R^*,R^h]}{\text{Var}[R^*]} \)

very general – but what is \( R^* \) in reality?
Simple CAPM with Quadratic Expected Utility

1. All agents are identical
   - Expected utility \( U(x_0, x_1) = \sum_s \pi_s u(x_0, x_s) \Rightarrow m = \frac{\partial_1 u}{E[\partial_0 u]} \)
   - Quadratic \( u(x_0, x_1) = v_0(x_0) - (x_1 - \alpha)^2 \)
     \[ \Rightarrow \partial_1 u = [-2(x_{1,1} - \alpha), \ldots, -2(x_{S,1} - \alpha)] \]
   - \( E[R^h] - R^f = -\frac{\text{Cov}[m, R^h]}{E[m]} \)
     \[ = -R^f \frac{\text{Cov}[\partial_1 u, R^h]}{E[\partial_0 u]} \]
     \[ = -R^f \frac{\text{Cov}[-2(x_1 - \alpha), R^h]}{E[\partial_0 u]} \]
     \[ = R^f 2\text{Cov}[x_1, R^h] / E[\partial_0 u] \]
   - Also holds for market portfolio
   - \( E[R^m] - R^f = R^f 2\text{Cov}[x_1, R^m] / E[\partial_0 u] \)

\[ \Rightarrow \frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{\text{Cov}[x_1, R^h]}{\text{Cov}[x_1, R^m]} \]
Simple CAPM with Quadratic Expected Utility

\[
\frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{\text{Cov}[x_1, R^h]}{\text{Cov}[x_1, R^m]}
\]

2. Homogenous agents + Exchange economy

⇒ \( x_1 = \text{agg. endowment and is perfectly correlated with } R^m \)

\[
\frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{\text{Cov}[R^m, R^h]}{\text{Var}[R^m]}
\]

since \( \beta^h = \frac{\text{Cov}[R^h, R^m]}{\text{Var}[R^m]} \)

\[E[R^h] = R^f + \beta^h \{E[R^m] - R^f\} \]

Market Security Line

**N.B.:** \( R^* = R^f \frac{(a+b_1 R^M)}{(a+b_1 R^f)} \) in this case (where \( b_1 < 0 \)!
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2. Mean-variance analysis
   – Portfolio Theory (Portfolio frontier, efficient frontier, …)
   – CAPM (Intuition)

3. CAPM
   – Projections
   – Pricing Kernel and Expectation Kernel
Definition: Mean-Variance Dominance & Efficient Frontier

- Asset (portfolio) A *mean-variance dominates* asset (portfolio) B if $\mu_A \geq \mu_B$ and $\sigma_A < \sigma_B$ or if $\mu_A > \mu_B$ while $\sigma_A \leq \sigma_B$.

- *Efficient frontier*: loci of all non-dominated portfolios in the mean-standard deviation space. By definition, no (“rational”) mean-variance investor would choose to hold a portfolio not located on the efficient frontier.
Expected Portfolio Returns & Variance

- **Expected returns (linear)**
  \[ \mu_p := E[r_p] = w_j \mu_j, \text{ where each } w_j = \frac{h_j}{\sum_j h_j} \]

- **Variance**
  \[ \sigma_p^2 := Var[r_p] = w'Vw = (w_1 \ w_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]
  \[ = (w_1 \sigma_1^2 + w_2 \sigma_{21} \ w_1 \sigma_{12} + w_2 \sigma_2^2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]
  \[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \geq 0 \]

  *since* \( \sigma_{12} \leq -\sigma_1 \sigma_2 \).  

  recall that correlation coefficient \( \in [-1,1] \)
Illustration of 2 Asset Case

- For certain weights: $w_1$ and $(1-w_1)$
  
  $\mu_p = w_1 \ E[r_1] + (1-w_1) \ E[r_2]$

  $\sigma^2_p = w_1^2 \ \sigma_1^2 + (1-w_1)^2 \ \sigma_2^2 + 2 \ w_1(1-w_1)\sigma_1 \ \sigma_2 \ \rho_{1,2}$

  (Specify $\sigma^2_p$ and one gets weights and $\mu_p$’s)

- Special cases [$w_1$ to obtain certain $\sigma_R$]
  
  - $\rho_{1,2} = 1 \Rightarrow w_1 = (+/-\sigma_p - \sigma_2) / (\sigma_1 - \sigma_2)$
  
  - $\rho_{1,2} = -1 \Rightarrow w_1 = (+/-\sigma_p + \sigma_2) / (\sigma_1 + \sigma_2)$
For $\rho_{1,2} = 1$: 

$$\sigma_p = |w_1 \sigma_1 + (1 - w_1) \sigma_2|$$

$$\mu_p = w_1 \mu_1 + (1 - w_1) \mu_2$$

Hence, 

$$w_1 = \frac{\pm \sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$$

$$\mu_p = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(\pm \sigma_p - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets

Lower part with … is irrelevant

$$\mu_p = E[r_1] + \frac{E[r_2] - E[r_1]}{\sigma_2 - \sigma_1}(-\sigma_R - \sigma_1)$$
For $\rho_{1,2} = -1$:

\[
\sigma_p = |w_1 \sigma_1 - (1 - w_1) \sigma_2| \\
\mu_p = w_1 \mu_1 + (1 - w_1) \mu_2
\]

Hence, $w_1 = \frac{\pm \sigma_p + \sigma_2}{\sigma_1 + \sigma_2}$

\[
\mu_p = \frac{\sigma_2}{\sigma_1 + \sigma_2} \mu_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2} \mu_2 \pm \frac{\mu_2 - \mu_1}{\sigma_1 + \sigma_2} \sigma_p
\]

Efficient Frontier: Two Perfectly Negative Correlated Risky Assets
For $-1 < \rho_{1,2} < 1$:

Efficient Frontier: Two Imperfectly Correlated Risky Assets
For $\sigma_1 = 0$

The Efficient Frontier: One Risky and One Risk Free Asset
Efficient frontier with n risky assets

- A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same expected portfolio return.

\[
\begin{align*}
\min_w & \quad \frac{1}{2} w^T V w \\
\text{s.t.} & \quad w^T e = E \\
\end{align*}
\]

\[
\begin{align*}
(\lambda) & \quad w^T 1 = 1 \\
(\gamma) & \quad \sum_{i=1}^{N} w_i E(\tilde{r}_i) = E \\
& \quad \sum_{i=1}^{N} w_i = 1
\end{align*}
\]
\[
\frac{\partial L}{\partial w} = Vw - \lambda e - \gamma 1 = 0
\]
\[
\frac{\partial L}{\partial \lambda} = E - w^T e = 0
\]
\[
\frac{\partial L}{\partial \gamma} = 1 - w^T 1 = 0
\]

The first FOC can be written as:

\[
Vw_p = \lambda e + \gamma 1 \quad \text{or}
\]
\[
w_p = \lambda V^{-1} e + \gamma V^{-1} 1
\]
\[
e^T w_p = \lambda (e^T V^{-1} e) + \gamma (e^T V^{-1} 1)
\]
Noting that $e^T w_p = w^T_p e$, using the first foc, the second foc can be written as

$$E[\tilde{r}_p] = e^T w_p = \lambda \left( e^T V^{-1} e \right) + \gamma \left( e^T V^{-1} 1 \right)$$

pre-multiplying first foc with 1 (instead of $e^T$) yields

$$1^T w_p = w^T_p 1 = \lambda \left( 1^T V^{-1} e \right) + \gamma \left( 1^T V^{-1} 1 \right) = 1$$

Solving both equations for $\lambda$ and $\gamma$

$$\lambda = \frac{CE - A}{D} \quad \text{and} \quad \gamma = \frac{B - AE}{D}$$

where $D = BC - A^2$. 
Hence, \( w_p = \lambda V^{-1}e + \gamma V^{-1}1 \) becomes

\[
w_p = \frac{CE - A}{D} V^{-1}e + \frac{B - AE}{D} V^{-1}1
\]

\((\text{vector}) (\text{vector})\)

\(\lambda \text{ (scalar)} \quad \gamma \text{ (scalar)}\)

\[
= \frac{1}{D} \left[ B(V^{-1}1) - A(V^{-1}e) \right] + \frac{1}{D} \left[ C(V^{-1}e) - A(V^{-1}1) \right] E
\]

**Result:** Portfolio weights are linear in expected portfolio return \( w_p = g + h \) \( E \)

If \( E = 0 \), \( w_p = g \)
If \( E = 1 \), \( w_p = g + h \)

Hence, \( g \) and \( g+h \) are portfolios on the frontier.
Characterization of Frontier Portfolios

• **Proposition 6.1**: The entire set of frontier portfolios can be generated by ("are convex combinations" of) \(g\) and \(g+h\).

• **Proposition 6.2**: The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios \(g\) and \(g+h\).

• **Proposition 6.3**: Any convex combination of frontier portfolios is also a frontier portfolio.
Characterization of Frontier Portfolios...

- For any portfolio on the frontier, \( \sigma^2(E[\tilde{r}_p]) = \left[g + hE(\tilde{r}_p)\right]^T V \left[g + hE(\tilde{r}_p)\right] \)

with \( g \) and \( h \) as defined earlier.

Multiplying all this out yields:

\[
\sigma^2(E[\tilde{r}_p]) = \frac{C}{D}[E[\tilde{r}_p] - \frac{A}{C}]^2 + \frac{1}{C}
\]
...Characterization of Frontier Portfolios...

• (i) the expected return of the minimum variance portfolio is $A/C$;

• (ii) the variance of the minimum variance portfolio is given by $1/C$;

• (iii) equation (6.17) is the equation of a parabola with vertex $(1/C, A/C)$ in the expected return/variance space and of a hyperbola in the expected return/standard deviation space. See Figures 6.3 and 6.4.
$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}} (\sigma^2 - \frac{1}{C})$$

Figure 6-3  The Set of Frontier Portfolios: Mean/Variance Space

Mean-Variance Analysis and CAPM
Figure 6-4  The Set of Frontier Portfolios: Mean/SD Space
Figure 6-5  The Set of Frontier Portfolios: Short Selling Allowed

Mean-Variance Analysis and CAPM
Efficient Frontier with risk-free asset

The Efficient Frontier: One Risk Free and n Risky Assets
Efficient Frontier with risk-free asset

\[
\begin{align*}
\min_w & \frac{1}{2} w^T V w \\
\text{s.t.} & \quad w^T e + (1 - w^T 1)r_f = E[r_p] \\
\text{FOC:} & \quad w_p = \lambda V^{-1}(e - r_f 1)
\end{align*}
\]

Multiplying by \((e-r_f 1)^T\) and solving for \(\lambda\) yields

\[
\lambda = \frac{E[r_p] - r_f}{(e-r_f 1)^T V^{-1} (e-r_f 1)}
\]

\[
w_p = V^{-1}(e - r_f 1) \left\{ \begin{array}{c} 1 \\ \frac{E[r_p] - r_f}{H^2} \end{array} \right\}
\]

where \(H = \sqrt{B - 2Ar_f + Cr_f^2}\)
Efficient frontier with risk-free asset

- **Result 1:** Excess return in frontier excess return

\[
Cov[r_q, r_p] = w_q^T V w_p
\]

\[
= w_q^T (e - r_f 1) \frac{E[r_p] - r_f}{H^2}
\]

\[
= \frac{(E[r_q] - r_f)(E[r_p] - r_f)}{H^2}
\]

\[
Var[r_p, r_p] = \frac{(E[r_p] - r_f)^2}{H^2}
\]

\[
E[r_q] - r_f = \frac{Cov[r_q, r_p]}{Var[r_p]} (E[r_p] - r_f)
\]

\[
: = \beta_{q,p}
\]

Holds for any frontier portfolio \( p \), in particular the market portfolio.
Efficient Frontier with risk-free asset

• Result 2: Frontier is linear in \((E[r], \sigma)\)-space

\[
Var[r_p, r_p] = \frac{(E[r_p] - r_f)^2}{H^2}
\]

\[
E[r_p] = r_f + H\sigma_p
\]

\[
H = \frac{E[r_p] - r_f}{\sigma_p}
\]

where \(H\) is the Sharpe ratio
Two Fund Separation

• Doing it in two steps:
  – First solve frontier for $n$ risky asset
  – Then solve tangency point

• Advantage:
  – Same portfolio of $n$ risky asset for different agents with different risk aversion
  – Useful for applying equilibrium argument (later)
Two Fund Separation

Price of Risk = highest Sharpe ratio

Optimal Portfolios of Two Investors with Different Risk Aversion
Mean-Variance Preferences

\[
\frac{\partial U}{\partial \mu_p} > 0, \quad \frac{\partial U}{\partial \sigma_p^2} < 0
\]

- \( U(\mu_p, \sigma_p) \) with
  \[
  E[W] - \frac{\gamma}{2} Var[W]
  \]
  - Example:

- Also in expected utility framework
  - quadratic utility function (with portfolio return \( R \))
    \[
    U(R) = a + b R + c R^2
    \]
    \[
    \nuNM: E[U(R)] = a + b E[R] + c E[R^2]
    = a + b \mu_p + c \mu_p^2 + c \sigma_p^2
    = g(\mu_p, \sigma_p)
    \]
  - asset returns normally distributed \( \Rightarrow R = \sum_j w^j r^j \) normal
    - if \( U(.) \) is CARA \( \Rightarrow \) certainty equivalent = \( \mu_p - \rho \lambda / 2\sigma_p^2 \)
      (Use moment generating function)
Equilibrium leads to CAPM

• Portfolio theory: only analysis of demand
  – price/returns are taken as given
  – composition of risky portfolio is same for all investors

• Equilibrium Demand = Supply (market portfolio)

• CAPM allows to derive
  – equilibrium prices/returns.
  – risk-premium
The CAPM with a risk-free bond

• The market portfolio is efficient since it is on the efficient frontier.
• All individual optimal portfolios are located on the half-line originating at point \((0,r_f)\).
• The slope of Capital Market Line (CML): \(\frac{E[R_M] - R_f}{\sigma_M}\).

\[
E[R_p] = R_f + \frac{E[R_M] - R_f}{\sigma_M} \sigma_p
\]
The Capital Market Line

Mean-Variance Analysis and CAPM
The Security Market Line

The Security Market Line (SML) is a graphical representation in finance that shows the relationship between expected returns and beta. The equation for the slope of the SML is:

\[ \text{slope SML} = \frac{E(r_i) - r_f}{\beta_i} \]

Where:
- \( E(r_i) \) is the expected return of security i
- \( r_f \) is the risk-free rate
- \( \beta_i \) is the beta of security i
- \( \beta_M = 1 \) is the beta of the market

The SML starts at the risk-free rate \( r_f \) on the horizontal axis and reaches the expected return of the market \( E(r_M) \) on the vertical axis.
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   - Portfolio Theory
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3. CAPM (modern derivation)
   - Projections
   - Pricing Kernel and Expectation Kernel
Projections

- States $s=1,\ldots,S$ with $\pi_s > 0$
- Probability inner product
  $$[x, y]_\pi = (xy)_\pi = \sum_s \pi_s x_s y_s = \sum_s (\sqrt{\pi_s} x_s \sqrt{\pi_s} y_s)$$
- $\pi$-norm $\|x\| = \sqrt{[x, x]_\pi}$ (measure of length)
  1. $\|x\| > 0 \ \forall x \neq 0$ and $\|x\| = 0$ if $x = 0$
  2. $\|\lambda x\| = |\lambda| \|x\|$
  3. $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in IR^S$
x and y are $\pi$-orthogonal iff $[x,y]_{\pi} = 0$, I.e. $E[xy] = 0$
...Projections...

- $\mathcal{Z}$ space of all linear combinations of vectors $z_1, \ldots, z_n$
- Given a vector $y \in \mathbb{R}^S$ solve
  $$
  \min_{\alpha \in \mathbb{R}^n} E[y - \sum_{j=1}^{\infty} \alpha^j z^j]^2
  $$
  FOC: (for each $j = 1, \ldots, n$)
  $$
  \sum S \pi_S (y_S - \sum_{j=1}^{\infty} \alpha^j z^j_S) z^j = 0
  $$
  $\Rightarrow$ $\hat{\alpha}$ the solution
  $$
  y^{Z} = \sum_{j=1}^{\infty} \hat{\alpha}^j z^j, \quad \epsilon := y - y^{Z}
  $$
- [smallest distance between vector $y$ and $\mathcal{Z}$ space]
...Projections

E[ε zi]=0 for each j=1,…,n (from FOC)
ε ⊥ z
y^Z is the (orthogonal) projection on Z
y = y^Z + ε’, y^Z ∈ Z, ε ⊥ z
Expected Value and Co-Variance...

squeeze axis by $\sqrt{\pi_s}$

\[ (1,1) \]

\[ [x,y] = E[xy] = \text{Cov}[x,y] + E[x]E[y] \]

\[ [x,x] = E[x^2] = \text{Var}[x] + E[x]^2 \]

\[ \|x\| = E[x^2]^{1/2} \]

\[ x = \hat{x} + \tilde{x} \]
...Expected Value and Co-Variance

\[ x = \hat{x} + \tilde{x}, \text{ where} \]
\[ \hat{x} \text{ is projection of } x \text{ onto } < 1 > \]
\[ \tilde{x} \text{ is projection of } x \text{ onto } < 1 > \perp \]

\[
E[x] = [x, 1]_\pi = [\hat{x}, 1]_\pi = \hat{x}[1, 1]_\pi = ||\hat{x}|| \\
Var[x] = [\tilde{x}, \tilde{x}]_\pi = E[\tilde{x}^2] = Var[\tilde{x}] \\
\sigma_x = ||\tilde{x}||_\pi = \text{standard deviation of } x \\
Cov[x, y] = Cov[\tilde{x}, \tilde{y}] = [\tilde{y}, \tilde{x}] \\
\]
Proof: \[ [x, y]_\pi = [\hat{x}, \tilde{y}]_\pi + [\tilde{x}, \tilde{y}]_\pi, \text{ since} \]
\[ [\tilde{y}, \tilde{x}]_\pi = [\tilde{y}, \hat{x}]_\pi = 0, [x, y]_\pi = E[\tilde{y}]E[\hat{x}] + Cov[\tilde{x}, \tilde{y}] \]
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New (LeRoy & Werner) Notation

• Main changes (new versus old)
  – gross return: $r = R$
  – SDF: $\mu = m$
  – pricing kernel: $k_q = m^*$
  – Asset span: $M = <X>$
  – income/endowment: $w_t = e_t$
Pricing Kernel $k_q$...

- $\mathcal{M}$ space of feasible payoffs.
- If no arbitrage and $\pi >>0$ there exists SDF $\mu \in \mathbb{R}^S$, $\mu >>0$, such that $q(z) = E(\mu z)$.
- $\mu \in \mathcal{M} – SDF$ need not be in asset span.
- A pricing kernel is a $k_q \in \mathcal{M}$ such that for each $z \in \mathcal{M}$, $q(z) = E(k_q z)$.
- $(k_q = m^* \text{ in our old notation.})$
Pricing Kernel - Examples...

- **Example 1:**
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0), x_2=(0,1,1), p=(1/3,2/3)$.
  - Then $k=(1,1,1)$ is the unique pricing kernel.

- **Example 2:**
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0), x_2=(0,1,0), p=(1/3,2/3)$.
  - Then $k=(1,2,0)$ is the unique pricing kernel.
...Pricing Kernel – Uniqueness

- If a state price density exists, there exists a unique pricing kernel.
  - If \( \dim(\mathcal{M}) = m \) (markets are complete), there are exactly \( m \) equations and \( m \) unknowns.
  - If \( \dim(\mathcal{M}) \leq m \), (markets may be incomplete).

For any state price density (SDF) \( \mu \) and any \( z \in \mathcal{M} \):

\[
E[(\mu - k_q)z] = 0
\]

\[\mu = (\mu - k_q) + k_q \Rightarrow k_q \text{ is the "projection" of } \mu \text{ on } \mathcal{M}.
\]

- Complete markets \( \Rightarrow k_q = \mu \) (SDF=state price density)
Expectations Kernel $k_e$

- An expectations kernel is a vector $k_e \in \mathcal{M}$
  - Such that $E(z) = E(k_e z)$ for each $z \in \mathcal{M}$.

- Example
  - $S=3$, $\pi_s=1/3$, for $s=1,2,3$, $x_1=(1,0,0)$, $x_2=(0,1,0)$.
  - Then the unique $k_e=(1,1,0)$.

- If $\pi >>0$, there exists a unique expectations kernel.
- Let $e=(1,\ldots, 1)$ then for any $z \in \mathcal{M}$
- $E[(e-k_e)z] = 0$
- $k_e$ is the “projection” of $e$ on $\mathcal{M}$
- $k_e = e$ if bond can be replicated (e.g. if markets are complete)
Mean Variance Frontier

• **Definition 1**: $z \in \mathcal{M}$ is in the mean variance frontier if there exists no $z' \in \mathcal{M}$ such that $E[z'] = E[z]$, $q(z') = q(z)$ and $\text{var}[z'] < \text{var}[z]$.

• **Definition 2**: Let $\mathcal{E}$ the space generated by $k_q$ and $k_e$.
  • Decompose $z = z^\mathcal{E} + \varepsilon$, with $z^\mathcal{E} \in \mathcal{E}$ and $\varepsilon \perp \mathcal{E}$.
  • Hence, $E[\varepsilon] = E[\varepsilon k_e] = 0$, $q(\varepsilon) = E[\varepsilon k_q] = 0$
  • $\text{Cov}[\varepsilon, z^\mathcal{E}] = E[\varepsilon z^\mathcal{E}] = 0$, since $\varepsilon \perp \mathcal{E}$.
  • $\text{var}[z] = \text{var}[z^\mathcal{E}] + \text{var}[\varepsilon]$ (price of $\varepsilon$ is zero, but positive variance)
  • If $z$ in mean variance frontier $\Rightarrow z \in \mathcal{E}$.
  • Every $z \in \mathcal{E}$ is in mean variance frontier.
Frontier Returns…

- Frontier returns are the returns of frontier payoffs with non-zero prices.

\[ r_e = \frac{k_e}{q(k_e)} = \frac{k_e}{E(k_q)} \]

\[ r_q = \frac{k_q}{q(k_q)} = \frac{k_q}{E(k_q k_q)} \]

- If \( z = \alpha k_q + \beta k_e \) then,

\[ r_z = \frac{\alpha q(k_q)}{\lambda} \frac{r_q}{\alpha q(k_q) + \beta q(k_e)} + \frac{\beta q(k_e)}{1 - \lambda} \frac{r_e}{\alpha q(k_q) + \beta q(k_e)} \]

- graphically: payoffs with price of \( p=1 \).
$$\mathcal{M} = R^S = R^3$$

Mean-Variance Payoff Frontier

Mean-Variance Return Frontier

$p=1$-line = return-line (orthogonal to $k_q$)
Mean-Variance Analysis and CAPM

NB: graphical illustrated of expected returns and standard deviation changes if bond is not in payoff span.
Mean-Variance (Payoff) Frontier

- Efficient (return) frontier
- Inefficient (return) frontier
- Standard deviation
- Expected return

Mean-Variance Analysis and CAPM
...Frontier Returns

If \( k_e = \alpha k_q \), frontier returns \( \equiv r_e \). (if agent is risk-neutral)

If \( k_e \neq \alpha k_q \), frontier can be written as:

\[
r_\lambda = r_e + \lambda (r_q - r_e)
\]

Expectations and Variance are

\[
E[r_\lambda] = E[r_e] + \lambda (E[r_q] - E[r_e])
\]

\[
\text{var}(r_\lambda) = \text{var}(r_e) + 2\lambda \text{cov}(r_e, r_q - r_e) + \lambda^2 \text{var}(r_q - r_e)
\]  

(1)

If risk-free asset exists, they simplify to:

\[
E[r_\lambda] = \bar{r} + \lambda (E[r_q] - \bar{r}).
\]

\[
\text{var}(r_\lambda) = \lambda^2 \text{var}(r_q). \quad \sigma(r_\lambda) = |\lambda| \sigma(r_q).
\]

\[
E(r_\lambda) = \bar{r} \pm \sigma(r_\lambda) \frac{E(r_q) - \bar{r}}{\sigma(r_q)}
\]
Minimum Variance Portfolio

• Take FOC w.r.t. $\lambda$ of
  \[ \text{var}(r_{\lambda}) = \text{var}(r_e) + 2\lambda \text{cov}(r_e, r_q - r_e) + \lambda^2 \text{var}(r_q - r_e) \]  
  (1)

• Hence, MVP has return of
  \[ r_e + \lambda_0 (r_q - r_e), \text{ with} \]
  \[ \lambda_0 = -\frac{\text{cov}(r_e, r_q - r_e)}{\text{var}(r_q - r_e)}. \]
Illustration of MVP

\[ M = R^2 \text{ and } S=3 \]

Expected return of MVP

Minimum standard deviation

Mean-Variance Analysis and CAPM
Mean-Variance Efficient Returns

- **Definition**: A return is **mean-variance efficient** if there is no other return with same variance but greater expectation.

- Mean variance efficient returns are frontier returns with $E[r_{\lambda}] \geq E[r_{\lambda,0}]$.

- If risk-free asset can be replicated
  - Mean variance efficient returns correspond to $\lambda \leq 0$.
  - Pricing kernel (portfolio) is not mean-variance efficient, since
    \[
    E[r_q] = \frac{E[k_q]}{E[k_q^2]} < \frac{1}{E[k_q]} = \bar{r}.
    \]
    Hint: $E[k_q^2] > E[k_q]^2$ since $Var[k_q] > 0$.
Zero-Covariance Frontier Returns

- Take two frontier portfolios with returns $r_\lambda = r_e + \lambda (r_q - r_e)$ and $r_\mu = r_e + \mu (r_q - r_e)$
- $\text{cov}(r_\mu, r_\lambda) = \text{var}(r_e) + (\lambda + \mu) \text{cov}(r_e, r_q - r_e) + \lambda \mu \text{var}(r_q - r_e)$
- The portfolios have zero co-variance if $\mu = -\frac{\text{var}(r_e) + \lambda \text{cov}(r_e, r_q - r_e)}{\text{cov}(r_e, r_q - r_e) + \lambda \text{var}(r_q - r_e)}$
- For all $\lambda \neq \lambda_0$ $\mu$ exists
- $\mu=0$ if risk-free bond can be replicated
Illustration of ZC Portfolio...

$\mathcal{M} = R^2$ and $S=3$

Recall:

$\text{COV}[x, y] = [\tilde{x}, \tilde{y}]_\pi$
Illustration of ZC Portfolio

Green lines do not necessarily cross.

(1,1,1)

\[ \|Z\bar{C}(p)\| \]

\[ \|\bar{p}\| \]

arbitrary portfolio \( p \)

ZC of \( p \)

Mean-Variance Analysis and CAPM
Beta Pricing...

- Frontier Returns (are on linear subspace). Hence
  \[ r_\beta = r_\mu + \beta (r_\lambda - r_\mu). \]
- Consider any asset with payoff \( x_j \)
  - It can be decomposed in \( x_j = x_j^\mathcal{E} + \varepsilon_j \)
  - \( q(x_j) = q(x_j^\mathcal{E}) \) and \( E[x_j] = E[x_j^\mathcal{E}] \), since \( \varepsilon \perp \mathcal{E} \).
  - Let \( r_j^\mathcal{E} \) be the return of \( x_j^\mathcal{E} \)
  - \( r_j = r_j^\mathcal{E} + \frac{\varepsilon_j}{q(x_j)}. \)
  - Using above and assuming \( \lambda \neq \lambda_0 \) and \( \mu \) is ZC-portfolio of \( \lambda \),
    \[ r_j = r_\mu + \beta_j (r_\lambda - r_\mu) + \frac{\varepsilon_j}{q(x_j)}. \]
**...Beta Pricing**

- Taking expectations and deriving covariance

\[ E[r_j] = E[r_\mu] + \beta_j (E[r_\lambda] - E[r_\mu]) \]

\[ \text{cov}(r_\lambda, r_j) = \beta_j \text{var}(r_\lambda) \Rightarrow \beta_j = \frac{\text{cov}(r_\lambda, r_j)}{\text{var}(r_\lambda)}. \]

- If risk-free asset can be replicated, beta-pricing equation simplifies to

\[ E[r_j] = \bar{r} + \beta_j (E[r_\lambda] - \bar{r}) \]

- Problem: How to identify frontier returns
Capital Asset Pricing Model...

- CAPM = market return is frontier return
  - Derive conditions under which market return is frontier return
  - Two periods: 0,1.
  - Endowment: individual $w_1^i$ at time 1, aggregate $\bar{w}_1 = \bar{w}_1^M + \bar{w}_1^N$, where $\bar{w}_1^M$ the orthogonal projection of $\bar{w}_1$ on $M$ is.
  - The market payoff: $m \equiv \bar{w}_1^M$
  - Assume $q(m) \neq 0$, let $r_m = m / q(m)$, and assume that $r_m$ is not the minimum variance return.
...Capital Asset Pricing Model

• If $r_{m0}$ is the frontier return that has zero covariance with $r_m$ then, for every security $j$,
  
  $E[r_j]=E[r_{m0}] + \beta_j (E[r_m]-E[r_{m0}])$, with
  
  $\beta_j = \frac{\text{cov}[r_j, r_m]}{\text{var}[r_m]}$.

• If a risk free asset exists, equation becomes,
  
  $E[r_j]=r_f + \beta_j (E[r_m]-r_f)$

• N.B. first equation always hold if there are only two assets.
Outdated material follows

• Traditional derivation of CAPM is less elegant
• Not relevant for exams
Characterization of Frontier Portfolios

- **Proposition 6.1**: The entire set of frontier portfolios can be generated by ("are convex combinations" of) \( g \) and \( g+h \).

- **Proposition 6.2**: The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios \( g \) and \( g+h \).

- **Proposition 6.3**: Any convex combination of frontier portfolios is also a frontier portfolio.
Characterization of Frontier Portfolios

• Proposition 6.1: The entire set of frontier portfolios can be generated by ("are convex combinations" of) \( g \) and \( g+h \).

  Proof: To see this let \( q \) be an arbitrary frontier portfolio with \( \bar{r}_q \) as its expected return. \( E(\bar{r}_q) \)

  Consider portfolio weights (proportions)

  \[ \pi_g = 1 - E(\bar{r}_q) \] and \( \pi_{g+h} = E(\bar{r}_q) \) then, as asserted,

  \[ [1 - E(\bar{r}_q)]g + E(\bar{r}_q)(g + h) = g + hE(\bar{r}_q) = w_q. \]
• **Proposition 6.2.** The portfolio frontier can be described as convex combinations of any **two** frontier portfolios, not just the frontier portfolios $g$ and $g+h$.

• **Proof:** To see this, let $p_1$ and $p_2$ be any two distinct frontier portfolios; since the frontier portfolios are $E[r_{p_1}] \neq E[r_{p_2}]$ different. Let $q$ be an arbitrary frontier portfolio, with expected return equal to $E[r_q]$. Since $E[r_{p_1}] \neq E[r_{p_2}]$, there must exist a unique number $\alpha$ such that

\[
E[r_q] = \alpha E[r_{p_1}] + (1 - \alpha) E[r_{p_2}] \tag{6.16}
\]

Now consider a portfolio of $p_1$ and $p_2$ with weights $\alpha$, $1 - \alpha$, respectively, as determined by (6.16). We must show that

\[
w_q = \alpha w_{p_1} + (1 - \alpha) w_{p_2}.
\]
Proof of Proposition 6.2 (continued)

\[ \alpha w_{p_1} + (1 - \alpha) w_{p_2} = \alpha [g + hE(\tilde{r}_{p_1})] + (1 - \alpha) [g + hE(\tilde{r}_{p_2})] \]

\[ = g + h[\alpha E(\tilde{r}_{p_1}) + (1 - \alpha) E(\tilde{r}_{p_2})] \]

\[ = g + hE(\tilde{r}_q), \text{ by construction} \]

\[ = w_q, \text{ since } q \text{ is a frontier portfolio.} \]
Proposition 6.3: Any convex combination of frontier portfolios is also a frontier portfolio.

Proof: Let \( \mathbf{\bar{w}} \) be a frontier portfolio, define \( N \) frontier portfolios (\( \mathbf{\bar{w}}_i \) represents the vector defining the composition of the \( i^{th} \) portfolio) and let \( \alpha_i \), \( i = 1, \ldots, N \) be real numbers such that \( \sum_{i=1}^{N} \alpha_i = 1 \). Lastly, let \( E(\mathbf{\bar{r}}_i) \) denote the expected return of portfolio with weights \( \mathbf{\bar{w}}_i \).

The weights corresponding to a linear combination of the above \( N \) portfolios are:

\[
\sum_{i=1}^{N} \alpha_i \mathbf{\bar{w}}_i = \sum_{i=1}^{N} \alpha_i (g + hE(\mathbf{\bar{r}}_i))
\]

\[
= \sum_{i=1}^{N} \alpha_i g + h \sum_{i=1}^{N} \alpha_i E(\mathbf{\bar{r}}_i)
\]

\[
= g + h \left[ \sum_{i=1}^{N} \alpha_i E(\mathbf{\bar{r}}_i) \right]
\]

Thus \( \sum_{i=1}^{N} \alpha_i \mathbf{\bar{w}}_i \) is a frontier portfolio with \( E(\mathbf{\bar{r}}) = \sum_{i=1}^{N} \alpha_i E(\mathbf{\bar{r}}_i) \).


Characterization of Frontier Portfolios...

- For any portfolio on the frontier, \( \sigma^2(E[\tilde{r}_p]) = [g + hE(\tilde{r}_p)]' V [g + hE(\tilde{r}_p)] \)
  with \( g \) and \( h \) as defined earlier.

Multiplying all this out yields:

\[
\sigma^2(E[\tilde{r}_p]) = \frac{C}{D} [E[\tilde{r}_p] - \frac{A}{C}]^2 + \frac{1}{C}
\]
Characterization of Frontier Portfolios...

(i) the expected return of the minimum variance portfolio is \( A/C \);

(ii) the variance of the minimum variance portfolio is given by \( 1/C \);

(iii) equation (6.17) is the equation of a parabola with vertex \((1/C, A/C)\) in the expected return/variance space and of a hyperbola in the expected return/standard deviation space. See Figures 6.3 and 6.4.
\[ E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}(\sigma^2 - \frac{1}{C}) \]
Figure 6-4  The Set of Frontier Portfolios: Mean/SD Space

Mean-Variance Analysis and CAPM
Figure 6-5  The Set of Frontier Portfolios: Short Selling Allowed

Mean-Variance Analysis and CAPM
Characterization of Efficient Portfolios (No Risk-Free Assets)

• **Definition 6.2:** *Efficient portfolios are those frontier portfolios which are not mean-variance dominated.*

• **Lemma:** *Efficient portfolios are those frontier portfolios for which the expected return exceeds A/C, the expected return of the minimum variance portfolio.*
• **Proposition 6.4:** *The set of efficient portfolios is a convex set.*
  - This does not mean, however, that the frontier of this set is convex-shaped in the risk-return space.

• **Proof:** Suppose each of the N portfolios considered above was efficient; then \( E(\bar{r}_i) \geq A / C \), for every portfolio \( i \).

However \( \sum_{i=1}^{N} \alpha_i E(\bar{r}_i) \geq \sum_{i=1}^{N} \alpha_i \frac{A}{C} = \frac{A}{C} \); thus, the convex combination is efficient as well. So the set of efficient portfolios, *as characterized by their portfolio weights*, is a convex set. ■
Zero Covariance Portfolio

• Zero-Cov Portfolio is useful for Zero-Beta CAPM

• Proposition 6.5: For any frontier portfolio $p$, except the minimum variance portfolio, there exists a unique frontier portfolio with which $p$ has zero covariance. We will call this portfolio the "zero covariance portfolio relative to $p"$, and denote its vector of portfolio weights by $ZC(p)$.

• Proof: by construction.
\[ \text{Cov}[r_p, r_q] := w_p^T V w_q \]
\[ \text{Cov}[r_p, r_q] = [\lambda V^{-1} e + \gamma V^{-1} 1]^T V w_q \]
\[ \text{Cov}[r_p, r_q] = \lambda e^T V^{-1} V w_q + \gamma 1^T V^{-1} V w_q \]
\[ \text{Cov}[r_p, r_q] = \lambda e^T w_q + \gamma \]
\[ \text{Cov}[r_p, r_q] = \lambda E[r_q] + \gamma \]

where \( \lambda = (CE[r_p] - A)/D \) and \( \gamma = (B - AE[r_p])/D \)

Hence,
\[ \text{Cov}[r_p, r_q] = \frac{CE[r_p] - A}{D} E[r_q] + \frac{B - AE[r_p]}{D} \]

collect all expected returns terms, add and subtract \( A^2C/DC^2 \)

and note that the remaining term \((1/C)((BC/D)-(A^2/D))=1/C\), since \( D=BC-A^2 \)

\[ \text{Cov}[r_p, r_q] = \frac{C}{D} [E[r_p] - \frac{A}{C}] [E[r_q] - \frac{A}{C}] + \frac{1}{C} \]
For zero co-variance portfolio ZC(p)

\[ \text{Cov}[r_p, r_{ZC(p)}] = 0 \]

\[ 0 = \frac{C}{D} \left[ \text{E}[r_p] - \frac{A}{C} \right] \left[ \text{E}[r_{ZC(p)}] - \frac{A}{C} \right] + \frac{1}{C} \]

\[ \text{E}[r_{ZC(p)}] = \frac{A}{C} - \frac{D/C^2}{\text{E}[r_p] - A/C} \]

For graphical illustration, let’s draw this line:

\[ \text{E}[r] = \frac{A}{C} - \frac{D/C^2}{\text{E}[r_p] - A/C} + \frac{\text{E}[r_p] - A/C}{\sigma^2[r_p] - 1/C} \sigma^2[r] \]
Graphical Representation:

\[
E[r] = \frac{A}{C} - \frac{D/C^2}{E[r_p] - A/C} + \frac{E[r_p] - A/C}{\sigma^2[r_p] - 1/C} \sigma^2[r]
\]

line through
\( \text{p} \) (Var\( [r_p] \), E\( [r_p] \)) AND
MVP (1/C, A/C)
(\text{use } \sigma^2(r) = \frac{C}{D} \left( E(\tilde{r}_p) - \frac{A}{C} \right)^2 + \frac{1}{C} \text{ )}

for \( \sigma^2(r) = 0 \) you get \( E[r_{ZC(p)}] \)
Figure 6-6  The Set of Frontier Portfolios: Location of the Zero-Covariance Portfolio
Zero-Beta CAPM
(no risk-free asset)

(i) agents maximize expected utility with increasing and strictly concave utility of money functions and asset returns are multivariate normally distributed, or

(ii) each agent chooses a portfolio with the objective of maximizing a derived utility function of the form $U(e, \sigma^2), U'_1 > 0, U'_2 < 0$, $U$ concave.

(iii) common time horizon,

(iv) homogeneous beliefs about $e$ and $\sigma^2$
– All investors hold mean-variance efficient portfolios

– the market portfolio is convex combination of efficient portfolios
  \[ \Rightarrow \text{ is efficient.} \]

– \( \text{Cov}[r_p, r_q] = \lambda \ E[r_q] + \gamma \) (q need not be on the frontier) (6.22)

– \( \text{Cov}[r_p, r_{ZC(p)}] = \lambda \ E[r_{ZC(p)}] + \gamma = 0 \)

\[
\text{Cov}[r_p, r_q] = \lambda \ \{E[r_q] - E[r_{ZC(p)}]\}
\]

\[
\text{Var}[r_p] = \lambda \ \{E[r_p] - E[r_{ZC(p)}]\}
\]

Divide third by fourth equation:

\[
E(\tilde{r}_q) = E(\tilde{r}_{ZC(M)}) + \beta_{Mq} \left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right]
\] (6.28)

\[
E(\tilde{r}_j) = E(\tilde{r}_{ZC(M)}) + \beta_{Mj} \left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right]
\] (6.29)
Zero-Beta CAPM

- **mean variance framework** (quadratic utility or normal returns)
- **In equilibrium, market portfolio, which is a convex combination of individual portfolios**

\[ E[r_q] = E[r_{ZC(M)}] + \beta_{Mq}[E[r_M]-E[r_{ZC(M)}]] \]

\[ E[r_j] = E[r_{ZC(M)}] + \beta_{Mj}[E[r_M]-E[r_{ZC(M)}]] \]