Information

• A (finite) set states $S$, probabilities $\pi_s > 0$ for each $s \in S$, and dates $t = 0, 1, \ldots, T$.

• At each date $t$ a collection of subsets of $S$, $F_t = \{A^1_t, A^2_t, \ldots, A^k_t\}$, such that $A^i_t \cap A^j_t = \emptyset$ if $j \neq i$ and $\bigcup_i A^i_t = S$. (Partition)
  
  - $F_0 = \{S\}$.
  
  - Each $A \in F_T$ contains exactly one state.
  
  - $A^i_t$, $i = 1, \ldots, k_t$, are the events at $t$.

• Information structure: $\mathcal{F} = \{F_0, F_1, \ldots, F_T\}$.

• Total recall: If $A \in F_t$, and $t' < t$ there exists an $A' \in F_{t'}$ such that $A \subset A'$.

• Trees
• Stochastic process: A collection of random variables \( y_t(s) \) for \( t = 0, \ldots, T \).

• Stochastic process is adapted to \( \mathcal{F} \) if for each \( A \in F_t, y_t(s) = y_t(s') \) for each \( s \in A \) and \( s' \in A \). \( y_t(A) \equiv y_t(s), s \in A \).

• \( y \geq 0 \) (positive) if \( y_t(s) \geq 0 \) for each \( (t, s) \). \( y > 0 \) (positive and non-zero) if \( y \geq 0 \) and \( y \neq 0 \). \( y \gg 0 \) (strictly positive) if \( y_t(s) > 0 \) for each \( (t, s) \).

• For each \( j = 1, \ldots, J \), a security is an adapted dividend process \( x^j_t, t = 1, \ldots, T \). The dividend paid at \( t \) is received by the agent that held the security from \( t - 1 \) to \( t \).

• The \((\text{ex dividend})\) price of this security is an adapted process \( p^j_t \).
Strategies

• A strategy consists of $J$ adapted processes $h_1^J, \ldots, h_J^J$. $h_t^j$ denotes the amount held from $t$ to $t+1$ of security $j$.

• $\mathcal{H}$ the set of all strategies.

• The dividend of the strategy is the process

$$z_t^h = (h_{t-1}^j - h_t^j) \cdot p_t + h_{t-1}^j \cdot x_t \equiv \sum_{j=1}^J [p_t^j (h_{t-1}^j - h_t^j) + h_{t-1}^j x_t^j], \text{ for } t \geq 1.$$ 

• $z^h$ is adapted.

• The cost of strategy $h$ is

$$z_0^h = p_0 \cdot h_0 \equiv \sum_j p_0^j h_0^j.$$
Marketed subspace

• $\mathcal{M}^p = \{y : y = z^h \text{ for some } h \in \mathcal{H}\}$.

• $\mathcal{M}^p$ is a linear space.

  – Complete markets if any adapted $y \in \mathcal{M}^p$.

  – Incomplete markets.
Dynamic hedging

- No dividends paid until period 2.
- Prices of the two assets in parentheses in periods 0, 1. Dividends in parentheses in period 2.

Hedge: $z_0 = 0$, $z_1 = 0$, $z_2 = (0,1,0,0)$
Arbitrage

• The law of one price holds if
\[ z^h = z^{h'} \implies p_0 \cdot h_0 = p_0 \cdot h'_0. \]

• Law of one price $\iff$ every portfolio strategy with zero payoff has zero price.

• If the law of one price holds we may define a linear functional $q : \mathcal{M}^p \to \mathbb{R}$ defined by:
\[ q(z) = p_0 \cdot h_0 \] for any strategy $h$ such that $z^h = z$. $q$ is the payoff pricing functional.

• A strong arbitrage is a strategy $h$ with $p_0 \cdot h_0 < 0$ and $z^h \geq 0$.

• An arbitrage is a strategy $h$ that is either a strong arbitrage or satisfies $p_0 \cdot h_0 = 0$ and $z^h > 0$. 
• The payoff pricing functional is strictly positive \((q(z) > 0 \text{ for every } z > 0, \ z \in M) \iff \text{there is no arbitrage.}\)

• The payoff pricing functional is positive \((q(z) \geq 0 \text{ for every } z \geq 0, \ z \in M) \iff \text{there is no strong arbitrage.}\)

• A one-period strong arbitrage at an event \(A^i_t\) at \(t < T\) is a portfolio \(h\) such that, for each \(s \in A^i_t\),

\[
[p_{t+1}(s) + x_{t+1}(s)] \cdot h \geq 0,
\]

and \(p_t(s) \cdot h < 0\).

• A one-period arbitrage at an event \(A^i_t\) at \(t < T\) is a portfolio \(h\) that is either a one-period strong arbitrage or that \(p_t(s) \cdot h = 0\) and \([p_{t+1}(s) + x_{t+1}(s)] \cdot h > 0\).

• No one-period arbitrage \(\iff\) no arbitrage.
Complete markets

• The immediate successors of an event $A \in F_t$ are all the events $B \in F_{t+1}$ such that $B \subset A$.

• $\iota(A) \equiv$ the number of immediate successors of an event $A$.

• The one-period payoff matrix in event $A \in F_t$, $t < T$, is the matrix with entries $p_{t+1}^j(B) + x_{t+1}^j(B)$, for $j = 1, \ldots J$ and $B$ an immediate successor of $A$.

• Markets are complete $\Leftrightarrow$ for all events $A \in F_t$, $t < T$ the one period payoff matrix has rank $\iota(A)$.

• $J \geq \iota(A_t^t)$ whenever $t < T$. 
Event prices

• Given an event $A_t \in F_t$, $t > 1$, consider a security that has a dividend process $y^{A_t}$ with $y^{A_t}_\tau(s) = 0$ if $\tau \neq t$, $y^{A_t}_t(s) = 0$ if $s \notin A_t$, and $y^{A_t}_t(s) = 1$ if $s \in A_t$.

• If complete markets and the law of one price prevails then $q(y^{A_t})$ is called the price of an elementary (Arrow-Debreu) claim associated with $A_t$, or the price of the event $A_t$.

• $q(y^{A_0}) = q(y^S) = 1$. 
• For any security \( x^j \), we may write:

\[
x^j = \sum_t \sum_{A_t \in F_t} x_t^j(A_t)y^A_t.
\]

•

\[
q(x^j) = \sum_{t=1}^T \sum_{A_t \in F_t} x_t^j(A_t)q(y^A_t). \tag{1}
\]

• For any strategy \( h \)

\[
p_0 \cdot h_0 = \sum_{t=1}^T \sum_{A_t \in F_t} z_t^h(A_t)q(y^A_t).
\]

• Suppose you follow strategy of buying an asset at \( t < T \) and selling at \( t + 1 \). Then:

\[
q(y^A_t)p^j(A_t) = \sum_{A_{t+1} \subseteq A_t, A_{t+1} \in F_{t+1}} q(y^{A_{t+1}})[p^j(A_{t+1}) + x^j(A_{t+1})].
\]
Implications of no arbitrage

• An event price vector is a family $q_t(A_t)$ for each $A_t \in F_t$, $0 \leq t \leq T$, with $q_0(S) = 1$.

• Event prices $q(A_t)$ are compatible with $(x, p)$ if for every $j = 1, \ldots, J$ and every $(t, A_t)$, $0 \leq t \leq T$ and $A_t \in F_t$

$$q_t(A_t)p^j(A_t) = \sum_{t<\tau\leq T} \sum_{A_\tau \subset A_t, A_\tau \in F_\tau} q_t(A_\tau)x^j(A_\tau). \quad (2)$$

• Equation (2) holds if and only if

$$q_t(A_t)p^j(A_t) = \sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} q_{t+1}(A_{t+1})[x^j(A_{t+1}) + p^j(A_{t+1})]$$
Two basic results

- There exists a strictly positive vector of event prices consistent with \((x, p)\) if and only if there is no arbitrage.

- If no arbitrage, markets are complete if and only if there exists a unique positive vector of event prices that are consistent with \((x, p)\).
Risk-free return and discount factors

• If $p^j_t(s) > 0$,

$$r^j_{t+1}(s) = \frac{p^j_{t+1}(s) + x^j_{t+1}(s)}{p^j_t(s)}$$

is the rate of return between $t$ and $t + 1$ in state $s$.

• $r^j_{t+1}(s)$ only depend on the event $A_{t+1} \in F_{t+1}$ that contains $s$.

• An asset $j$ is risk free at $(t, s)$ if $r^j_{t+1}(s)$ only depends on the event $A_t \in F_t$ that contains $s$. 
• If no arbitrage, all risk-free assets at a given 
\((t, s)\) must have the same \(\bar{r}_t(s) > 0\).

• If \(A_t \in F_t\), let \(\bar{r}_t(A_t) = \bar{r}_t(s)\), for any \(s \in A_t\).

• 
\[
q_t(A_t) = \bar{r}_{t+1}(A_{t+1}) \sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} q_{t+1}(A_{t+1})
\]

• If a risk-free security exists at each \((t, s)\), 
the discount factor between zero and \(t\) in 
state \(s\) is \(\rho_0(s) = 1\) and, 
\[
\rho_t(s) = \prod_{\tau=1}^t (\bar{r}_\tau(s))^{-1}, \ t \geq 1
\]

– \(\rho_t(s)\) only depends on the event \(A_{t-1} \in F_{t-1}\) that contains \(s\).

– \(\rho_t(s) = \rho_{t+1}(s)\bar{r}_{t+1}(s)\)

• If \(A_t \in F_t\) let \(\rho_t(A_t) = \rho_t(s)\) for any \(s \in A_t\).
An example: model of interest rate

- State $s = (r_0, r_1, \ldots, r_T)$, where $r_t$ is one plus the one period interest rate prevailing at $t$.

- A one year bond issued at $t$ pays a dividend of 1 at $t + 1$. The price at $t$ is $\frac{1}{r_t(s)}$.

- A $\tau$ year zero coupon bond issued at $t$ pays a dividend of 1 at $t + \tau$.

  - A $\tau$ year zero coupon issued at $t$ is a $\tau - j$ bond in period $t + j$.

- $P^{\tau}_t(s)$ the price of a $\tau$ year zero coupon bond as of $t$, $(P^{0}_t(s)=1)$. If $A_t \in F_t$,

$$
\sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} P^{\tau}_t(A_t)q_t(A_t) = P^{\tau-1}_{t+1}(A_{t+1})q_{t+1}(A_{t+1}).
$$
Risk-neutral probabilities

• Assume no arbitrage or equivalently that you have a set of positive state prices, and that at each \((t, s)\) a one period risk free security exists.

• Recall that \(\{s\} \in F_T\), and set \(q(s) = q_T(\{s\})\).

• Let

\[
\pi^*(s) = \frac{q(s)}{\rho_T(s)} > 0,
\]

• For every \(A \subset S\) let

\[
\pi^*(A) = \sum_{s \in A} \pi^*(s).
\]
\begin{itemize}
  \item If $A_{T-1} \in F_{T-1}$, then 
    \[ q_{T-1}(A_{T-1}) = \bar{r}_T(s) \sum_{s \in A_{T-1}} q(s) \]

  \item \[ q_{T-1}(A_{T-1}) = \frac{\rho_{T-1}(s)}{\rho_T(s)} \sum_{s \in A_{T-1}} q(s) = \]
    \[ \rho_{T-1}(A_{T-1}) \sum_{s \in A_{T-1}} \frac{q(s)}{\rho_T(s)} = \rho_{T-1}(A_{T-1}) \pi^*(A_{T-1}) \]

  \item \[ \pi^*(A_{T-1}) = \frac{q_{T-1}(A_{T-1})}{\rho_{T-1}(A_{T-1})} \]

  \item If $A_t \in F_t$ then 
    \[ \pi^*(A_t) = \frac{q_t(A_t)}{\rho_t(A_t)} \]

  \item \[ \pi^*(S) = \frac{q_0(S)}{\rho_0(S)} = 1 \]

  \item $\pi^*$ is a probability.
\end{itemize}
• Let $A_t \in F_t$ and $A_{t+1} \subset A_t$, $A_{t+1} \in F_{t+1}$.

$\pi^*(A_{t+1}|A_t) \equiv \frac{\pi^*(A_{t+1})}{\pi^*(A_t)}$

$\pi^*(A_{t+1}|A_t) = \frac{q_{t+1}(A_{t+1})}{q_t(A_t)}\bar{r}_{t+1}(A_{t+1})$.

$p^j(A_t) = [\bar{r}_{t+1}(A_{t+1})]^{-1} \times \sum \pi^*(A_{t+1}|A_t)[p^j_{t+1}(A_{t+1}) + x^j_{t+1}(A_{t+1})]$, where the sum is over all $A_{t+1} \subset A_t$ and $A_{t+1} \in F_{t+1}$. 
All securities have same expected returns under $\pi^*$.

Prices equal discounted expected payoffs.

Risk-neutral probabilities.
Black-Scholes model

• $\rho_t = (\bar{r})^{-t}$. If $A_t$ is event in which there are exactly $\ell$ downs between 0 and $t$,

$$q_t(A_t) = \left( \frac{u - \bar{r}}{\bar{r}(u - d)} \right)^{\ell} \left( \frac{\bar{r} - d}{\bar{r}(u - d)} \right)^{t - \ell},$$

•

$$\pi^*(A_t) = \left( \frac{u - \bar{r}}{u - d} \right)^{\ell} \left( \frac{\bar{r} - d}{u - d} \right)^{t - \ell}.$$

• European call option with strike $K$: No dividend before $T$ and if $\ell$ downs occur until $T$, dividend at $T$ is $(u^{T - \ell}d^\ell - K)^+$.  

• Price of option

$$\sum_{\ell=0}^{T} \binom{T}{\ell} \frac{1}{(\bar{r})^T} (u^{T - \ell}d^\ell - K)^+ \left( \frac{u - \bar{r}}{u - d} \right)^{\ell} \left( \frac{\bar{r} - d}{u - d} \right)^{T - \ell}. $$
A model for interest rates

• $s = (r_0, r_1, \ldots, r_T)$, where $r_t$ is one plus the one period interest rate prevailing at $t$. Conditional on the interest rate prevailing today, there are two possible interest rates in the next period $i_u > i_d > 0$.

• The short rate volatility at $t$ in state $s$ is:

$$
\sigma_t(s) = \frac{\ln \left( \frac{i_u}{i_d} \right)}{2}
$$

• $\sigma_t(s)$ equals the standard deviation of the log of the short rate that will prevail tomorrow, assuming that the probability that $i_{t+1} = i_u$ conditional on today’s rate is .5.

• Probabilities are the risk-neutral probabilities.
• Example: \( i_0 = 0.04 \), the price of a two period bond \( p^2 = 0.925 \), and \( \sigma_0 = 10\% \).

•

\[
ru = rd e^{2\sigma_0}.
\]

•

\[
p_2 = \left( \frac{1}{1 + i_0} \right) \frac{1}{2} \left[ \frac{1}{1 + i_u} + \frac{1}{1 + i_d} \right]
\]

• Two equations on two unknowns. Solution \( r_d = 0.0356, ru = 0.0435 \).

• Suppose \( i_{ud} = i_{du} \), and you know the price of a three period bond \( p^3 \). Still need to determine 3 rates and you have only one equation.
• One possible solution:

\[ \sigma_1(i_u) = \sigma_1(i_d) = \sigma_1, \text{ given.} \]

\[ \frac{i_{uu}}{i_{ud}} = \frac{i_{du}}{i_{dd}}. \]

\[ i_{ud} = \sqrt{i_{uu}i_{dd}}. \]

• Works for any \( T \).
Security gains as martingales

• The gains process of security $j$ is the process given by:

$$g^j_t(s) = p^j_t(s) + \sum_{\tau=1}^{t} \frac{\rho_{\tau}(s)}{\rho_t(s)} x^j_{\tau}(s)$$

• The discounted gain process for security $j$ is:

$$d^j_t(s) = \rho_t(s)g^j_t(s) = \rho_t(s)p^j_t(s) + \sum_{\tau=1}^{t} \rho_{\tau}(s)x^j_{\tau}(s)$$

•

$$d^j_{t+1}(s) - d^j_t(s) = \rho_{t+1}(s)(x^j_{t+1}(s) + p^j_{t+1}(s)) - \rho_t(s)p^j_t(s).$$
Multiplying equation (3) by $\rho_t(s)$, we obtain:

$$E_t^*[\rho_{t+1}(s)(x^j_{t+1}(s) + p^j_{t+1}(s))] = \rho_t(s)p^j_t(s).$$

Hence:

$$E_t^*[d^j_{t+1}(s) - d^j_t(s)] = 0.$$

$$E_t^*[d^j_{t+1}] = d^j_t.$$

If $\tau > t$,

$$E_t^*[d^j_{\tau}] = d^j_t.$$

d$^j$ is a martingale.

Also works for portfolios of securities.
Forwards

• A payoff $W(s)$ at time $T$.

• A forward contract is an agreement struck at $t < T$ to pay an amount $F_t$, the forward price at $T$ in exchange for $W$. Nothing else is exchanged.

\[
0 = E_t^* \left[ \rho_T (F_t - W) \right]
\]

• If the discount factor is non-random,

\[
F_t = E_t^* [W].
\]

– $F_t$ is a martingale.

– Short rate of interest is deterministic.
Futures

• A futures price process is a process $\phi_t$ with $\phi_T = W$. A futures contract has a dividend stream that equals $\phi_t - \phi_{t-1}$, for each $t \geq 1$. The price of a futures contract is always zero.

• $0 = E^*[\phi_{t+1} - \phi_t]$.

• $\phi_t = E^*[\phi_{t+1}]$.

• $\phi_t = E^*[W]$.

• Future price process is always a martingale.
The pricing kernel

- The pricing kernel is a process $k \in \mathcal{M}^p$ such that for any $z \in \mathcal{M}^p$,

$$q(z) = \sum_{t=1}^{T} E[k_t z_t]$$

- If $s \in A_t$ let $k_t(A_t) \equiv k_t(s)$.

- If complete markets,

$$k_t(A_t) = \frac{q_t(A_t)}{\pi(A_t)}$$

- Applying this equation to the strategy “buy security $j$ at $t$ in every $s \in A_t$ and sell it at $t + 1$” we obtain:

$$k_t p_t^j = E_t[k_{t+1}(p_{t+1}^j + x_{t+1}^j)]$$
\[ k_t = E_t[k_{t+1}r_{t+1}^j] \]

\[ k_t = \bar{r}_{t+1}E_t[k_{t+1}]. \]

\[ E_t[k_{t+1}(x_{t+1}^j + p_{t+1}^j - \bar{r}_{t+1}p_t^j)] = 0. \]

The process \( \tilde{g}^j \) defined as:

\[ \tilde{g}^j_t(s) = g^j_t(s)k_t(s), \]

is a martingale.

Also works for portfolios.
Utility maximization

• The consumption process $c$ an adapted process (sometimes assumed non-negative.)

• The utility function $u$ associating to each $c$ a real number.

• A set of tradeable assets and an endowment process $w$.

•

$$\max_{c,h} u(c), \text{ subject to }$$

$$c_0 = w_0 - p_0 \cdot h_0 \text{ and } c_t = w_t + z_t^h.$$
• If we have an interior solution then at any state $s \in A_t$

$$p(A_t) = \sum [p(A_{t+1}) + x(A_{t+1})] \frac{\partial U}{\partial c_t(A_t)} \frac{\partial c_{t+1}(A_{t+1})}{\partial U},$$

where the sum is over all $A_{t+1} \subset A_t$, and $A_{t+1} \in F_{t+1}$.

• If complete markets prevail then the only constraint is $q \cdot c = q \cdot w$.

•

$$q_t(A_t) = \frac{\partial U}{\partial c_t(A_t)} \frac{\partial U}{\partial c_0}.$$