Lecture 09: Multi-period Model
Fixed Income, Futures, Swaps

Prof. Markus K. Brunnermeier
Overview

1. Bond basics
2. Duration
3. Term structure of the real interest rate
4. Forwards and futures
   1. Forwards versus futures prices
   2. Currency futures
   3. Commodity futures: backwardation and contango
5. Repos
6. Swaps
Bond basics

• Example: U.S. Treasury (Table 7.1)
  - Bills (<1 year), no coupons, sell at discount
  - Notes (1-10 years), Bonds (10-30 years), coupons, sell at par
  - STRIPS: claim to a single coupon or principal, zero-coupon

• Notation:
  - $r_t(t_1, t_2)$: Interest rate from time $t_1$ to $t_2$ prevailing at time $t$.
  - $P_{t_0}(t_1, t_2)$: Price of a bond quoted at $t= t_0$ to be purchased at $t= t_1$ maturing at $t= t_2$
  - Yield to maturity: Percentage increase in $s$ earned from the bond
Bond basics (cont.)

• Zero-coupon bonds make a single payment at maturity

![Table 7.1](image)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Zero-Coupon Bond Yield</th>
<th>Zero-Coupon Bond Price</th>
<th>One-Year Implied Forward Rate</th>
<th>Par Coupon</th>
<th>Continuously Compounded Zero Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.00%</td>
<td>0.943396</td>
<td>6.00000%</td>
<td>6.00000%</td>
<td>5.82689%</td>
</tr>
<tr>
<td>2</td>
<td>6.50</td>
<td>0.881659</td>
<td>7.00236</td>
<td>6.48423</td>
<td>6.29748</td>
</tr>
<tr>
<td>3</td>
<td>7.00</td>
<td>0.816298</td>
<td>8.00705</td>
<td>6.95485</td>
<td>6.76586</td>
</tr>
</tbody>
</table>

➢ One year zero-coupon bond: $P(0,1)=0.943396$
  • Pay $0.943396$ today to receive $1$ at $t=1$
  • Yield to maturity (YTM) $= 1/0.943396 - 1 = 0.06 = 6\% = r(0,1)$

➢ Two year zero-coupon bond: $P(0,2)=0.881659$
  • YTM$=1/0.881659 - 1=0.134225=(1+r(0,2))^2=>r(0,2)=0.065=6.5\%$
Bond basics (cont.)

- Zero-coupon bond price that pays $C_t$ at $t$:
- Yield curve: Graph of annualized bond yields against time
  \[ P(0,t) = \frac{C_t}{[1 + r(0,t)]^t} \]
- Implied forward rates
  - Suppose current one-year rate $r(0,1)$ and two-year rate $r(0,2)$
  - Current forward rate from year 1 to year 2, $r_0(1,2)$, must satisfy:
    \[ [1+r_0(0,1)] [1+r_0(1,2)] = [1+r_0(0,2)]^2 \]
Bond basics (cont.)

- Earn $r(0, 1)$
- Earn implied forward rate, $r(1, 2)$
- Earn $r(0, 2)$ per year

\[
[1 + r(0, 1)] \times [1 + r(1, 2)]
\]

\[
[1 + r(0, 2)]^2
\]
Bond basics (cont.)

- In general: 
  \[ [1 + r_0(t_1, t_2)]^{t_2-t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)} \]

- Example 7.1:
  - What are the implied forward rate \( r_0(2,3) \) and forward zero-coupon bond price \( P_0(2,3) \) from year 2 to year 3? (use Table 7.1)

  \[
  r_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.0800705
  \]

  \[
  P_0(2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.816298}{0.881659} = 0.925865
  \]
Coupon bonds

- The price at time of issue of $t$ of a bond maturing at time $T$ that pays $n$ coupons of size $c$ and maturity payment of $\$1$:

$$B_t(t,T,c,n) = \sum_{i=1}^{n} cP_t(t,t_i) + P_t(t,T)$$

where $t_i = t + i(T - t)/n$

- For the bond to sell at par, i.e. $B(t,T,c,n) = 1$ the coupon size must be:

$$c = \frac{1 - P_t(t,T)}{\sum_{i=1}^{n} P_t(t,t_i)}$$
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Duration

- Duration is a measure of sensitivity of a bond’s price to changes in interest rates

  - **Duration**
    
    \[
    \frac{\text{Change in bond price}}{\text{Unit change in yield}} = -\frac{1}{1 + y} \sum_{i=1}^{n} \frac{C_i}{(1 + y)^i}
    \]
    
    Divide by 100 (1,000) for change in price given a 1% (1 basis point) change in yield

  - **Modified Duration**
    
    % Change in price for a unit change in yield
    
    \[
    - \text{Duration} \times \frac{1}{B(y)} = \frac{1}{1 + y} \sum_{i=1}^{n} \left[ \frac{C_i}{(1 + y)^i} \frac{1}{B(y)} \right]
    \]

  - **Macaulay Duration**
    
    Size-weighted average of time until payments
    
    \[
    \text{Duration} \times \frac{1 + y}{B(y)} = \sum_{i=1}^{n} \left[ \frac{C_i}{(1 + y)^i} \frac{1}{B(y)} \right]
    \]

- \( y \): yield per period;
  to annualize divide by \# of payments per year

- \( B(y) \): bond price as a function of yield \( y \)
Duration (Examples)

- **Example 7.4 & 7.5**
  - 3-year zero-coupon bond with maturity value of $100
    - Bond price at YTM of 7.00%: \( \frac{100}{(1.0700^3)} = 81.62979 \) $81.62979
    - Bond price at YTM of 7.01%: \( \frac{100}{(1.0701^3)} = 81.60691 \) $81.60691
    - Duration:
      \[
      - \frac{1}{1.07} \times 3 \times \frac{100}{1.07^3} = -828.87
      \]
    - For a basis point (0.01%) change: \(-828.87/10,000=-0.02289\)
    - Macaulay duration:
      \[
      -(-828.87) \times \frac{1.07}{81.62979} = 3.000
      \]
- **Example 7.6**
  - 3-year annual coupon (6.95485%) par bond
    - Macaulay Duration:
      \[
      (1 \times \frac{0.0695485}{1.0695485}) + (2 \times \frac{0.0695485}{1.0695485^2}) + (3 \times \frac{1.0695485}{1.0695485^3}) = 2.80915
      \]
Duration (cont.)

• What is the new bond price \( B(y+\varepsilon) \) given a small change \( \varepsilon \) in yield?
  
  - Rewrite the Macaulay duration:
    \[
    D_{Mac} = - \frac{[B(y + \varepsilon) - B(y)]}{\varepsilon} \frac{1 + y}{B(y)}
    \]
  
  - And rearrange:
    \[
    B(y + \varepsilon) = B(y) - [D_{Mac} \times \frac{B(y)\varepsilon}{1 + y}]
    \]

• Example 7.7
  
  - Consider the 3-year zero-coupon bond with price $81.63 and yield 7%
  
  - What will be the price of the bond if the yield were to increase to 7.25%?
    \[
    B(7.25\%) = $81.63 - (3 \times $81.63 \times 0.0025 / 1.07) = $81.058
    \]
  
  - Using ordinary bond pricing: \( B(7.25\%) = $100 / (1.0725)^3 = $81.060 \)

• The formula is only approximate due to the bond’s convexity
Duration matching

• Suppose we own a bond with time to maturity $t_1$, price $B_1$, and Macaulay duration $D_1$
• How many ($N$) of another bond with time to maturity $t_2$, price $B_2$, and Macaulay duration $D_2$ do we need to short to eliminate sensitivity to interest rate changes? The hedge ratio:

$$N = -\frac{D_1 B_1(y_1) / (1 + y_1)}{D_2 B_2(y_2) / (1 + y_2)}$$

➢ Using $B(y + \epsilon) = B(y) - [D_{Mac} \times \frac{B(y) \epsilon}{1 + y}]$ for portfolio $B_1 + NB_2$
• The value of the resulting portfolio with duration zero is $B_1 + NB_2$
• Example 7.8
  ➢ We own a 7-year 6% annual coupon bond yielding 7%
  ➢ Want to match its duration by shorting a 10-year, 8% bond yielding 7.5%
  ➢ You can verify that $B_1 = $94.611, $B_2 = $103.432, $D_1 = 5.882$, and $D_2 = 7.297$

$$N = -\frac{5.882 \times 94.611 / 1.07}{7.297 \times 103.432 / 1.075} = -0.7409$$
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Term structure of real interest rates

- Bond prices carry all the information on intertemporal rates of substitution,
  - primarily affected by expectations, and
  - only indirectly by risk considerations.

- Collection of interest rates for different times to maturity is a meaningful predictor of future economic developments.
  - More optimistic expectations produce an upward-sloping term structure of interest rates.
Term structure

- The price of a risk-free discount bond which matures in period $t$ is $\beta_t = E[M_t] = 1/(1+y_t)^t$
- The corresponding (gross) yield is $1+y_t = (\beta_t)^{-1/t} = \delta^{-1} \left[ E[u'(w_t)] / u'(w_0) \right]^{-1/t}$.
- Collection of interest rates is the term structure, $(y_1, y_2, y_3, \ldots)$.
- Note that these are real yield rates (net of inflation), as are all prices and returns.
Term structure

• Left figure is an example of the term structure of **real** interest rates, measured with U.S. Treasury Inflation Protected Securities (TIPS), on August 2, 2004.  

• Right hand shows nominal yield curve  
  **Source**: [www.bloomberg.com](http://www.bloomberg.com)
Term structure

\[ 1 + y_t = \delta^{-1} \left\{ \frac{E[u'(w_t)]}{u'(w_0)} \right\}^{-1/t}. \]

- Let \( g_t \) be the (state dependent) growth rate per period between period \( t \) and period \( 0 \), so \((1 + g_t)^t = w_t / w_0\).
- Assume further that the representative agent has CRRA utility and a first-order approximations yields
  \[ y_t \approx \gamma E\{g_t\} - \ln \delta. \]  
  (Homework! Note \( \ln(1+y) \approx y \))
- The yield curve measures expected growth rates over different horizons.
Term structure

\[ y_t \approx \gamma \ E\{g_t\} - \ln \delta \]

- Approximation ignores second-order effects of uncertainty
- …but we know that more uncertainty depresses interest rates if the representative agent is prudent.
- Thus, if long horizon uncertainty about the per capita growth rate is smaller than about short horizons (for instance if growth rates are mean reverting), then the term structure of interest rates will be upward sloping.
The expectations hypothesis

- **cross section of prices:**
The term structure are bond prices *at a particular point in time*. This is a cross section of prices.

- **time series properties:**
  how do interest rates evolve as time goes by?

- Time series view is the relevant view for an investor how tries to decide what kind of bonds to invest into, or what kind of loan to take.
The expectations hypothesis

• Suppose you have some spare capital that you will not need for 2 years.

• You could invest it into 2 year discount bonds, yielding a return rate of $y_{0,2}$.

• Of course, since bonds are continuously traded, you could alternatively invest into 1-year discount bonds, and then roll over these bonds when they mature. The expected yield is $(1+y_{0,1})E[(1+y_{1,2})]$. 

• Or you could buy a 3-year bond and sell it after 2 years.

• Which of these possibilities is the best?
The expectations hypothesis

• Only the first strategy is truly free of risk.

• The other two strategies are risky, since
  ➢ price of 3-year bond in period 2 is unknown today, &
  ➢ tomorrow's yield of a 1-year bond is not known today.

• Term premia:
  the possible premium that these risky strategies
  have over the risk-free strategy are called *term premia*. (special form of risk premium).
The expectations hypothesis

- Consider a $t$-period discount bond. The price of this bond
  \[ \beta_{0,t} = \mathbb{E}[M_t] = \mathbb{E}[m_1 \cdots m_t]. \]
  - one has to invest $\beta_{0,t}$ in $t=0$ in order to receive one consumption unit in period $t$.

- Alternatively, one could buy 1-period discount bonds and roll them over $t$-times. The investment that is necessary today to get one consumption unit (in expectation) in period $t$ with this strategy is
  \[ \mathbb{E}[m_1] \cdots \mathbb{E}[m_t]. \]

(to see this for $t=2$: buying $\beta_{1,2}$ bonds at $t=0$ costing $\beta_{0,1} \beta_{1,2}$ pays in expectations at $t=1$, which allows to pay one bond which ultimately pays $1$ at $t=2$)
The expectations hypothesis

- Two strategies yield same expected return rate if and only if
  \[ E[m_1 \cdots m_t] = E[m_1] \cdots E[m_t], \]
  which holds if \( m_t \) is serially uncorrelated.

  - In that case, there are no term premia — an assumption known as the expectations hypothesis.
  - Whenever \( m_t \) is serially correlated (for instance because the growth process is serially correlated), then expectations hypothesis may fail.
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Futures contracts

• Exchange-traded “forward contracts”
• Typical features of futures contracts
  ➢ Standardized, specified delivery dates, locations, procedures
  ➢ A clearinghouse
    • Matches buy and sell orders
    • Keeps track of members’ obligations and payments
    • After matching the trades, becomes counterparty
• Differences from forward contracts
  ➢ Settled daily through mark-to-market process ➔ low credit risk
  ➢ Highly liquid ➔ easier to offset an existing position
  ➢ Highly standardized structure ➔ harder to customize
Example: S&P 500 Futures

- WSJ listing:

<table>
<thead>
<tr>
<th></th>
<th>OPEN</th>
<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHANGE</th>
<th>LIFETIME HIGH</th>
<th>LIFETIME LOW</th>
<th>OPEN INT.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500 Index (CME)-$250 times Index</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>112350</td>
<td>112370</td>
<td>109100</td>
<td>109530</td>
<td>-</td>
<td>2810</td>
<td>134960</td>
<td>94100</td>
</tr>
<tr>
<td>June</td>
<td>111950</td>
<td>111950</td>
<td>109350</td>
<td>109730</td>
<td>-</td>
<td>2830</td>
<td>170550</td>
<td>95030</td>
</tr>
<tr>
<td>Dec</td>
<td>111580</td>
<td>111580</td>
<td>110020</td>
<td>110390</td>
<td>-</td>
<td>2930</td>
<td>150070</td>
<td>96130</td>
</tr>
<tr>
<td>Est vol</td>
<td>79,914</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>vol Fri</td>
<td>65,250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>open int</td>
<td>502,626</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-701</td>
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<tr>
<td>idx prf.</td>
<td>Hi 1122.20</td>
<td>Lo 1092.25</td>
<td>Close 1094.44</td>
<td>-27.76</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Contract specifications:

<table>
<thead>
<tr>
<th>Specifications for the S&amp;P 500 index futures contract.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underlying</td>
</tr>
<tr>
<td>Where traded</td>
</tr>
<tr>
<td>Size</td>
</tr>
<tr>
<td>Months</td>
</tr>
<tr>
<td>Trading ends</td>
</tr>
<tr>
<td>Settlement</td>
</tr>
</tbody>
</table>
Example: S&P 500 Futures (cont.)

- Notional value: $250 x Index
- Cash-settled contract
- Open interest: total number of buy/sell pairs
- Margin and mark-to-market
  - Initial margin
  - Maintenance margin (70-80% of initial margin)
  - Margin call
  - Daily mark-to-market
- Futures prices vs. forward prices
  - The difference negligible especially for short-lived contracts
  - Can be significant for long-lived contracts and/or when interest rates are correlated with the price of the underlying asset
Example: S&P 500 Futures (cont.)

- Mark-to-market proceeds and margin balance for 8 long futures:

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier ($)</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000.00</td>
<td>1100.00</td>
<td>—</td>
<td>220,000.00</td>
</tr>
<tr>
<td>1</td>
<td>2000.00</td>
<td>1027.99</td>
<td>−72.01</td>
<td>76,233.99</td>
</tr>
<tr>
<td>2</td>
<td>2000.00</td>
<td>1037.88</td>
<td>9.89</td>
<td>96,102.01</td>
</tr>
<tr>
<td>3</td>
<td>2000.00</td>
<td>1073.23</td>
<td>35.35</td>
<td>166,912.96</td>
</tr>
<tr>
<td>4</td>
<td>2000.00</td>
<td>1048.78</td>
<td>−24.45</td>
<td>118,205.66</td>
</tr>
<tr>
<td>5</td>
<td>2000.00</td>
<td>1090.32</td>
<td>41.54</td>
<td>201,422.13</td>
</tr>
<tr>
<td>6</td>
<td>2000.00</td>
<td>1106.94</td>
<td>16.62</td>
<td>234,894.67</td>
</tr>
<tr>
<td>7</td>
<td>2000.00</td>
<td>1110.98</td>
<td>4.04</td>
<td>243,245.86</td>
</tr>
<tr>
<td>8</td>
<td>2000.00</td>
<td>1024.74</td>
<td>−86.24</td>
<td>71,046.69</td>
</tr>
<tr>
<td>9</td>
<td>2000.00</td>
<td>1007.30</td>
<td>−17.44</td>
<td>36,248.72</td>
</tr>
<tr>
<td>10</td>
<td>2000.00</td>
<td>1011.65</td>
<td>4.35</td>
<td>44,990.57</td>
</tr>
</tbody>
</table>

Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The final row represents expiration of the contract.
Forwards versus futures pricing

• Price of Forward using EMM is
  
  \[ 0 = E_t^*[\rho_T (F_{0,T} - S_T)] = E_t^*[\rho_T] (F_{0,T} - E_t^*[S_T]) - \text{Cov}_t^*[\rho_T, S_T] \]
  for fixed interest rate
  
  \[ F_{0,T} = E_t^*[S_T] \]

• Price of Futures contract is always zero.
  Each period there is a “dividend” stream of \( \phi_t - \phi_{t-1} \)
  and \( \phi_T = S_T \)
  
  \[ 0 = E_t^*[\rho_{t+1}(\phi_{t+1} - \phi_t)] \] for all t
  since \( \rho_{t+1} \) is known at t

  \[ \phi_t = E_t^*[\phi_{t+1}] \] and \( \phi_T = S_T \)
  
  \[ \phi_t = E_t^*[S_T] \]
  Futures price process is always a martingale
Example: S&P 500 Futures (cont.)

- S&P index arbitrage: comparison of formula prices with actual prices:

<table>
<thead>
<tr>
<th>Expiration Month:</th>
<th>March</th>
<th>June</th>
<th>December</th>
</tr>
</thead>
<tbody>
<tr>
<td>Days to Expiration:</td>
<td>42</td>
<td>140</td>
<td>322</td>
</tr>
<tr>
<td>S&amp;P 500 Index Futures Price</td>
<td>1130.4</td>
<td>1132.5</td>
<td>1140.3</td>
</tr>
<tr>
<td>Treasury-Bill Yield</td>
<td>0.0167</td>
<td>0.017</td>
<td>0.0218</td>
</tr>
<tr>
<td>Theoretical Forward Price</td>
<td>1130.68</td>
<td>1131.93</td>
<td>1139.01</td>
</tr>
<tr>
<td>LIBOR</td>
<td>0.0187</td>
<td>0.0201</td>
<td>0.0240</td>
</tr>
<tr>
<td>Theoretical Forward Price</td>
<td>1130.94</td>
<td>1133.28</td>
<td>1141.22</td>
</tr>
</tbody>
</table>

S&P 500 index futures prices and interest rate information from the Wall Street Journal, February 1, 2002. The closing S&P 500 spot price was 1130.20. Treasury-bill yields are reported yields on Treasury bills expiring in the same month as the futures contract. LIBOR rates are constructed from Eurodollar prices. The theoretical forward prices are constructed for each maturity from equation (5.7) using the interest rate in the preceding row and assuming a 1.3% dividend yield.
Uses of index futures

- Why buy an index futures contract instead of synthesizing it using the stocks in the index? Lower transaction costs
- Asset allocation: switching investments among asset classes
- Example: Invested in the S&P 500 index and temporarily wish to temporarily invest in bonds instead of index. What to do?
  - Alternative #1: Sell all 500 stocks and invest in bonds
  - Alternative #2: Take a short forward position in S&P 500 index

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Own Stock @ $100</td>
<td>Today: $-100, 1 year, S₁ = $80: $80, 1 year, S₁ = $130: $130</td>
</tr>
<tr>
<td>Short Forward @ $110</td>
<td>Today: 0, 1 year, S₁ = $80: $110 - $80, 1 year, S₁ = $130: $110 - $130</td>
</tr>
<tr>
<td>Total</td>
<td>Today: $-100, 1 year, S₁ = $80: $110, 1 year, S₁ = $130: $110</td>
</tr>
</tbody>
</table>

**Table 5.10**

Effect of owning the stock and selling forward, assuming that $S₀ = $100$ and $F₀,₁ = $110$. 
Uses of index futures (cont.)

• $100 million portfolio with $\beta$ of 1.4 and $r_f = 6\%$

1. Adjust for difference in $\$\$\$ amount
   • 1 futures contract $250 \times 1100 = \$275,000$
   • Number of contracts needed $\$100\text{mill}/\$0.275\text{mill} = 363.636$

2. Adjust for difference in $\beta$
   $$363.636 \times 1.4 = 509.09\text{ contracts}$$
Uses of index futures (cont.)

- Cross-hedging with perfect correlation

<table>
<thead>
<tr>
<th>S&amp;P 500 Index</th>
<th>Gain on 509 Futures</th>
<th>Portfolio Value</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>33.855</td>
<td>72.145</td>
<td>106.000</td>
</tr>
<tr>
<td>950</td>
<td>27.491</td>
<td>78.509</td>
<td>106.000</td>
</tr>
<tr>
<td>1000</td>
<td>21.127</td>
<td>84.873</td>
<td>106.000</td>
</tr>
<tr>
<td>1050</td>
<td>14.764</td>
<td>91.236</td>
<td>106.000</td>
</tr>
<tr>
<td>1100</td>
<td>8.400</td>
<td>97.600</td>
<td>106.000</td>
</tr>
<tr>
<td>1150</td>
<td>2.036</td>
<td>103.964</td>
<td>106.000</td>
</tr>
<tr>
<td>1200</td>
<td>-4.327</td>
<td>110.327</td>
<td>106.000</td>
</tr>
</tbody>
</table>

- Cross-hedging with imperfect correlation
- General asset allocation: futures overlay
- Risk management for stock-pickers
Currency contracts

- Widely used to hedge against changes in exchange rates
- WSJ listing:

<table>
<thead>
<tr>
<th>CURRENCY</th>
<th>OPEN</th>
<th>HIGH</th>
<th>LOW</th>
<th>SETTLE</th>
<th>CHANGE</th>
<th>LIFETIME</th>
<th>OPEN INT.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Japan Yen (CME)-12.5 million yen; $ per yen (.00)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.7528</td>
<td>.7600</td>
<td>.7508</td>
<td>.7576</td>
<td>+ .0050</td>
<td>.8760</td>
<td>.7416</td>
</tr>
<tr>
<td>June</td>
<td>.7570</td>
<td>.7635</td>
<td>.7547</td>
<td>.7611</td>
<td>+ .0050</td>
<td>.8776</td>
<td>.7453</td>
</tr>
<tr>
<td>Est vol</td>
<td>11,091; vol Fri 25,220; open int 126,860</td>
<td>-1,352.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Canadian Dollar (CME)-100,000 drs.; $ per Can $</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.6282</td>
<td>.6287</td>
<td>.6284</td>
<td>.6266</td>
<td>- .0016</td>
<td>.6725</td>
<td>.6170</td>
</tr>
<tr>
<td>June</td>
<td>.6280</td>
<td>.6287</td>
<td>.6263</td>
<td>.6264</td>
<td>- .0016</td>
<td>.6700</td>
<td>.6180</td>
</tr>
<tr>
<td>Sept</td>
<td>.6266</td>
<td>.6262</td>
<td>.6259</td>
<td>.6266</td>
<td>- .0016</td>
<td>.6590</td>
<td>.6175</td>
</tr>
<tr>
<td>Dec</td>
<td>.6274</td>
<td>.6260</td>
<td>.6255</td>
<td>.6269</td>
<td>- .0016</td>
<td>.6555</td>
<td>.6190</td>
</tr>
<tr>
<td>Est vol</td>
<td>5,343; vol Fri 7,699; open int 60,818</td>
<td>-652.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>British Pound (CME)-62,500 pds.; $ per pound</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>1.4112</td>
<td>1.4206</td>
<td>1.4106</td>
<td>1.4186</td>
<td>+ .0058</td>
<td>1.4700</td>
<td>1.3810</td>
</tr>
<tr>
<td>June</td>
<td>1.4066</td>
<td>1.4140</td>
<td>1.4038</td>
<td>1.4102</td>
<td>+ .0058</td>
<td>1.4550</td>
<td>1.3910</td>
</tr>
<tr>
<td>Est vol</td>
<td>5,315; vol Fri 4,859; open int 34,067</td>
<td>-174.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Swiss Franc (CME)-125,000 francs; $ per franc</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.5836</td>
<td>.5902</td>
<td>.5825</td>
<td>.5892</td>
<td>+ .0061</td>
<td>.6370</td>
<td>.5540</td>
</tr>
<tr>
<td>June</td>
<td>.5866</td>
<td>.5906</td>
<td>.5830</td>
<td>.5895</td>
<td>+ .0061</td>
<td>.6320</td>
<td>.5813</td>
</tr>
<tr>
<td>Est vol</td>
<td>5,676; vol Fri 6,330; open int 47,871</td>
<td>-587.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Australian Dollar (CME)-100,000 drs.; $ per A.$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.5076</td>
<td>.5102</td>
<td>.5074</td>
<td>.5098</td>
<td>+ .0031</td>
<td>.5300</td>
<td>.4810</td>
</tr>
<tr>
<td>June</td>
<td>.5050</td>
<td>.5099</td>
<td>.5046</td>
<td>.5070</td>
<td>+ .0031</td>
<td>.5218</td>
<td>.4885</td>
</tr>
<tr>
<td>Est vol</td>
<td>944; vol Fri 1,871; open int 22,079</td>
<td>-518.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Mexican Peso (CME)-500,000 new Mex. peso, $ per MP</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.10843</td>
<td>.10850</td>
<td>.10810</td>
<td>.10835</td>
<td>+ 000005</td>
<td>.10940</td>
<td>.09770</td>
</tr>
<tr>
<td>June</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sept</td>
<td>.10450</td>
<td>.10450</td>
<td>.10450</td>
<td>.10453</td>
<td>+ 0000010</td>
<td>.10750</td>
<td>.09730</td>
</tr>
<tr>
<td>Dec</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Est vol</td>
<td>1,940; vol Fri 2,817; open int 30,163</td>
<td>+626.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Euro FX (CME)-Euro 125,000; $ per Euro</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>.8601</td>
<td>.8694</td>
<td>.8593</td>
<td>.8666</td>
<td>+ .0085</td>
<td>.9630</td>
<td>.8336</td>
</tr>
<tr>
<td>June</td>
<td>.8663</td>
<td>.8663</td>
<td>.8554</td>
<td>.8657</td>
<td>+ .0085</td>
<td>.9275</td>
<td>.8365</td>
</tr>
<tr>
<td>Dec</td>
<td>.8601</td>
<td>.8600</td>
<td>.8600</td>
<td>.8616</td>
<td>+ .0085</td>
<td>.9175</td>
<td>.8390</td>
</tr>
<tr>
<td>Est vol</td>
<td>14,904; vol Fri 17,547; open int 105,729</td>
<td>-560.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Currency contracts: pricing

• Currency prepaid forward
  ➢ Suppose you want to purchase ¥1 one year from today using $s
  ➢ \( F_{0,T}^P = x_0 e^{-r_y T} \) (price of prepaid forward)
    • where \( x_0 \) is current ($/¥) exchange rate, and \( r_y \) is the yen-denominated interest rate
    • Why? By deferring delivery of the currency one loses interest income from bonds denominated in that currency

• Currency forward
  ➢ \( F_{0,T} = x_0 e^{(r-r_y)T} \)
    • \( r \) is the $-denominated domestic interest rate
    • \( F_{0,T} > x_0 \) if \( r > r_y \) (domestic risk-free rate exceeds foreign risk-free rate)
Currency contracts: pricing (cont.)

- Example 5.3:
  - ¥-denominated interest rate is 2% and current ($/¥) exchange rate is 0.009. To have ¥1 in one year one needs to invest today:
    - \( 0.009 / ¥ \times ¥1 \times e^{-0.02} = $0.008822 \)

- Example 5.4:
  - ¥-denominated interest rate is 2% and $-denominated rate is 6%. The current ($/¥) exchange rate is 0.009. The 1-year forward rate:
    - \( 0.009e^{0.06-0.02} = 0.009367 \)
Currency contracts: pricing (cont.)

- Synthetic currency forward: borrowing in one currency and lending in another creates the same cash flow as a forward contract.
- Covered interest arbitrage: offset the synthetic forward position with an actual forward contract.

**Table 5.12**

Synthetically creating a yen forward contract by borrowing in dollars and lending in yen. The payoff at time 1 is ¥1 – $0.009367.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Year 0</th>
<th>Year 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Borrow $x_0 e^{-r_y} Dollar at 6% ($)</td>
<td>+0.008822</td>
<td></td>
</tr>
<tr>
<td>Convert to Yen @ 0.009 $/¥</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Invest in Yen-Denominated Bill (¥)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>-0.009367</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
</tr>
<tr>
<td>Year 0</td>
</tr>
<tr>
<td>$0.008822</td>
</tr>
<tr>
<td>-0.008822</td>
</tr>
<tr>
<td>—</td>
</tr>
<tr>
<td>—</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-0.009367</td>
</tr>
</tbody>
</table>
Eurodollar futures

- **WSJ listing**

- **Contract specifications**

  ![Figure 5.7](image)

  Specifications for the Eurodollar futures contract.

  Where traded  Chicaco Mercantile Exchange

  Size  3-month Eurodollar time deposit, $1 million principal

  Months  Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month

  Trading ends  5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.

  Delivery  Cash settlement

  Settlement  100 — British Banker’s Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)

---

Princeton University
Introduction to Commodity Forwards

- *Commodity* forward prices can be described by the same formula as that for *financial* forward prices:

\[ F_{0,T} = S_0 e^{(r-\delta)T} \]

- For financial assets, \( \delta \) is the dividend yield. For commodities, \( \delta \) is the commodity lease rate. The lease rate is the return that makes an investor willing to buy and then lend a commodity.

- The lease rate for a commodity can typically be estimated only by observing the forward prices.
Introduction to Commodity Forwards

- The set of prices for different expiration dates for a given commodity is called the **forward curve** (or the **forward strip**) for that date.

- If on a given date the forward curve is upward-sloping, then the market is in **contango**. If the forward curve is downward sloping, the market is in **backwardation**.

  - Note that forward curves can have portions in backwardation and portions in contango.
**Forward rate agreements**

- FRAs are over-the-counter contracts that guarantee a borrowing or lending rate on a given notional principal amount.
- Can be settled at maturity (in arrears) or the initiation of the borrowing or lending transaction.
  - FRA settlement in arrears: \((r_{qrtly} - r_{FRA}) \times \text{notional principal}\)
  - At the time of borrowing: \(\text{notional principal} \times \frac{r_{qrtly} - r_{FRA}}{1 + r_{qrtly}}\)
- FRAs can be synthetically replicated using zero-coupon bonds.
Forward rate agreements (cont.)

**TABLE 7.2**

Investment strategy undertaken at time 0, resulting in net cash flows of −$1 on day $t$, and receiving the implied forward rate, $1 + r_0(t, t + s)$ at $t + s$. This synthetically creates the cash flows from entering into a forward rate agreement on day 0 to lend at day $t$.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0</strong></td>
<td>0</td>
</tr>
<tr>
<td><strong>$t$</strong></td>
<td>$1 + r_0(t, t + s)$</td>
</tr>
<tr>
<td><strong>$t + s$</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Buy $1 + r_0(t, t + s)$ Zeros</strong></td>
<td>$-P(0, t + s) \times (1 + r_0(t, t + s))$</td>
</tr>
<tr>
<td>Maturing at $t + s$</td>
<td>1</td>
</tr>
<tr>
<td><strong>Short 1 Zero Maturing at $t$</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
</tr>
</tbody>
</table>

**TABLE 7.3**

Example of synthetic FRA. The transactions in this table are exactly those in Table 7.2, except that all bonds are sold at time $t$.

<table>
<thead>
<tr>
<th>Transaction</th>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0</strong></td>
<td>0</td>
</tr>
<tr>
<td><strong>$t$</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Buy $1 + r_0(t, t + s)$ Zeros</strong></td>
<td>$-P(0, t + s) \times [1 + r_0(t, t + s)]$</td>
</tr>
<tr>
<td>Maturing at $t + s$</td>
<td>$1 + r_0(t, t + s)$</td>
</tr>
<tr>
<td><strong>Short 1 Zero Maturing at $t$</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
</tr>
</tbody>
</table>

**Note:**
- $P(0, t + s)$ is the price of a zero-coupon bond with maturity $t + s$.
- $r_0(t, t + s)$ is the forward rate from time $t$ to $t + s$.
Eurodollar futures

- Very similar in nature to an FRA with subtle differences
  - The settlement structure of Eurodollar contracts favors borrowers
  - Therefore the rate implicit in Eurodollar futures is greater than the FRA rate => Convexity bias
- The payoff at expiration: \([\text{Futures price} - (100 - r_{\text{LIBOR}})] \times 100 \times \$25\)
- Example: Hedging $100 million borrowing with Eurodollar futures:

<table>
<thead>
<tr>
<th>TABLE 7.4</th>
<th>Results from hedging $100m in borrowing with 98.23 short Eurodollar futures.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Borrowing Rate:</strong></td>
<td><strong>Cash Flows</strong></td>
</tr>
<tr>
<td>Borrow $100m</td>
<td>1.5%</td>
</tr>
<tr>
<td>Gain on 98.23 Short Eurodollar Contracts</td>
<td>0.294695m</td>
</tr>
<tr>
<td>Gain Plus Interest</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Eurodollar futures (cont.)

- Recently Eurodollar futures took over T-bill futures as the preferred contract to manage interest rate risk.
- **LIBOR** tracks the corporate borrowing rates better than the T-bill rate.
Interest rate strips and stacks

- Suppose we will borrow $100 million in 6 months for a period of 2 years by rolling over the total every 3 months

  \[ r_1=? \quad r_2=? \quad r_3=? \quad r_4=? \quad r_5=? \quad r_6=? \quad r_7=? \quad r_8=? \]

  \[ \begin{array}{cccccccc}
  0 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
  $100 & $100 & $100 & $100 & $100 & $100 & $100 & $100 & $100 \\
  \end{array} \]

- Two alternatives to hedge the interest rate risk:
  - Strip: Eight separate $100 million FRAs for each 3-month period
  - Stack: Enter 6-month FRA for ~$800 million. Each quarter enter into another FRA decreasing the total by ~$100 each time

  Strip is the best alternative but requires the existence of FRA far into the future. Stack is more feasible but suffers from basis risk
Treasury bond/note futures

- WSJ listings for T-bond and T-note futures

**INTEREST RATE**

Treasu ry Bonds (CBT)-$100,000; pts 32nds of 100%
Mar 103-11 104-08 103-11 104-05 + 28 111-16 97-11 457,263
June 102-17 103-02 102-14 103-00 + 26 110-00 96-30 41,817
Est vol 122,000; vol Fri 193,638; open int 499,310, +10,880.

Treas u ry Notes (CBT)-$100,000; pts 32nds of 100%
Mar 105-09 106-31 106-08 06-265 + 20,5 11-085 101-23 532,323
June 105-07 05-205 105-04 105-19 + 20,5 107-10 101-10 61,297
Est vol 220,000; vol Fri 272,886; open int 593,529, +1,178.

10 Yr Agency Notes (CBT)-$100,000; pts 32nds of 100%
Mar 102-04 102-14 102-01 102-13 + 18,5 106-04 96-27 36,659
Est vol 1,000; vol Fri 1,519; open int 36,658, +68.

5 Yr Treasury Notes (CBT)-$100,000; pts 32nds of 100%
Mar 06-145 105-27 06-115 103-25 + 12,5 109-08 04-015 531,946
June 105-29 105-29 05-205 105-29 + 14,0 06-105 103-30 30,569
Est vol 107,000; vol Fri 152,108; open int 592,515, +2,338.

2 Yr Treasury Notes (CBT)-$200,000; pts 32nds of 100%
Mar 04-282 105-02 04-282 105-02 + 6,5 105-27 03-255 104,180
Est vol 6,500; vol Fri 11,586; open int 104,180, +858.

**FIGURE 7.6**

Specifications for the Treasury-note futures contract.

<table>
<thead>
<tr>
<th>Where traded</th>
<th>CBOT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underlying</td>
<td>6% 10-year Treasury note</td>
</tr>
<tr>
<td>Size</td>
<td>$100,000 Treasury note</td>
</tr>
<tr>
<td>Months</td>
<td>Mar, Jun, Sep, Dec, out 15 months</td>
</tr>
<tr>
<td>Trading ends</td>
<td>Seventh business day preceding last business day of month. Delivery until last business day of month.</td>
</tr>
<tr>
<td>Delivery</td>
<td>Physical T-note with at least 6.5 years to maturity and not more than 10 years to maturity. Price paid to the short for notes with other than 6% coupon is determined by multiplying futures price by a conversion factor. The conversion factor is the price of the delivered note ($1 par value) to yield 6%. Settlement until last business day of the month.</td>
</tr>
</tbody>
</table>
Treasury bond/note futures (cont.)

- Long T-note futures position is an obligation to buy a 6% bond with maturity between 6.5 and 10 years to maturity
- The short party is able to choose from various maturities and coupons: the “cheapest-to-deliver” bond
- In exchange for the delivery the long pays the short the “invoice price.”

\[
\text{Invoice price} = (\text{Futures price} \times \text{conversion factor}) + \text{accrued interest}
\]

### Table 7.5

<table>
<thead>
<tr>
<th>Description</th>
<th>8-Year 7% Coupon, 6.4% Yield</th>
<th>7-Year 5%, 6.3% Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price</td>
<td>103.71</td>
<td>92.73</td>
</tr>
<tr>
<td>Price at 6% (Conversion Factor)</td>
<td>106.28</td>
<td>94.35</td>
</tr>
<tr>
<td>Invoice Price (Futures \times Conversion Factor)</td>
<td>103.71</td>
<td>92.09</td>
</tr>
<tr>
<td>Invoice – Market</td>
<td>0</td>
<td>–0.66</td>
</tr>
</tbody>
</table>

Prices, yields, and the conversion factor for two bonds. The futures price is 97.583. The short would break even delivering the 8-year 7% bond, and lose money delivering the 7-year 5% bond. Both bonds make semiannual coupon payments.
Overview

1. Bond basics
2. Duration
3. Term structure of the real interest rate
4. Forwards and futures
   - Forwards versus futures price
   - Currency futures
   - Commodity futures: backwardation and contango
5. Repos
6. Swaps
Repurchase agreements

- A repurchase agreement or a repo entails selling a security with an agreement to buy it back at a fixed price.
- The underlying security is held as collateral by the counterparty => A repo is collateralized borrowing.
- Used by securities dealers to finance inventory.
- A “haircut” is charged by the counterparty to account for credit risk.
Overview

1. Bond basics
2. Duration
3. Term structure of the real interest rate
4. Forwards and futures
   - Forwards versus futures price
   - Currency futures
   - Commodity futures: backwardation and contango
5. Repos
6. Swaps
Introduction to Swaps

• A swap is a contract calling for an exchange of payments, on one or more dates, determined by the difference in two prices.

• A swap provides a means to hedge a stream of risky payments.

• A single-payment swap is the same thing as a cash-settled forward contract.
An example of a commodity swap

• An industrial producer, IP Inc., needs to buy 100,000 barrels of oil 1 year from today and 2 years from today.
• The forward prices for deliver in 1 year and 2 years are $20 and $21/barrel.
• The 1- and 2-year zero-coupon bond yields are 6% and 6.5%.
An example of a commodity swap

- IP can guarantee the cost of buying oil for the next 2 years by entering into long forward contracts for 100,000 barrels in each of the next 2 years. The PV of this cost per barrel is

$$\frac{20}{1.06} + \frac{21}{1.065^2} = 37.383$$

- Thus, IP could pay an oil supplier $37.383, and the supplier would commit to delivering one barrel in each of the next two years.
An example of a commodity swap

- With a prepaid swap, the buyer might worry about the resulting credit risk. Therefore, a better solution is to defer payments until the oil is delivered, while still fixing the total price.

\[ \frac{x}{1.06} + \frac{x}{1.065^2} = \$37.383 \]

- Any payment stream with a PV of $37.383 is acceptable. Typically, a swap will call for equal payments in each year.

  ➢ For example, the payment per year per barrel, \( x \), will have to be $20.483 to satisfy the following equation:
Physical versus financial settlement

- **Physical settlement** of the swap:

  ![Diagram showing physical settlement of a swap]

  - **Oil Buyer**
  - **Swap Counterparty**
  - **$20.483**
  - **Oil**
Physical versus financial settlement

- **Financial settlement** of the swap:

  - The oil buyer, IP, pays the swap counterparty the difference between $20.483 and the spot price, and the oil buyer then buys oil at the spot price.

  - If the difference between $20.483 and the spot price is negative, then the swap counterparty pays the buyer.
Physical versus financial settlement

- The results for the buyer are the same whether the swap is settled physically or financially. In both cases, the net cost to the oil buyer is $20.483.
• Swaps are nothing more than forward contracts coupled with borrowing and lending money.

➢ Consider the swap price of $20.483/barrel. Relative to the forward curve price of $20 in 1 year and $21 in 2 years, we are overpaying by $0.483 in the first year, and we are underpaying by $0.517 in the second year.

➢ Thus, by entering into the swap, we are lending the counterparty money for 1 year. The interest rate on this loan is

\[
\frac{0.517}{0.483} - 1 = 7\%.
\]

➢ Given 1- and 2-year zero-coupon bond yields of 6% and 6.5%, 7% is the 1-year implied forward yield from year 1 to year 2.
The market value of a swap

- The market value of a swap is zero at interception.
- Once the swap is struck, its market value will generally no longer be zero because:
  - the forward prices for oil and interest rates will change over time;
  - even if prices do not change, the market value of swaps will change over time due to the implicit borrowing and lending.
- A buyer wishing to exit the swap could enter into an offsetting swap with the original counterparty or whomever offers the best price.
- The market value of the swap is the difference in the PV of payments between the original and new swap rates.
Interest Rate Swaps

• The notional principle of the swap is the amount on which the interest payments are based.
• The life of the swap is the swap term or swap tenor.
• If swap payments are made at the end of the period (when interest is due), the swap is said to be settled in arrears.
An example of an interest rate swap

- XYZ Corp. has $200M of floating-rate debt at LIBOR, i.e., every year it pays that year’s current LIBOR.
- XYZ would prefer to have fixed-rate debt with 3 years to maturity.
- XYZ could enter a swap, in which they receive a floating rate and pay the fixed rate, which is 6.9548%.
An example of an interest rate swap

On net, XYZ pays 6.9548%:

\[
\text{XYZ net payment} = -\text{LIBOR} + \text{LIBOR} - 6.9548\% = -6.9548\%
\]

Floating Payment       Swap Payment
Computing the swap rate

- Suppose there are $n$ swap settlements, occurring on dates $t_i$, $i = 1, \ldots, n$.
- The implied forward interest rate from date $t_{i-1}$ to date $t_i$, known at date 0, is $r_0(t_{i-1}, t_i)$.
- The price of a zero-coupon bond maturing on date $t_i$ is $P(0, t_i)$.
- The fixed swap rate is $R$.

- The market-maker is a counterparty to the swap in order to earn fees, not to take on interest rate risk. Therefore, the market-maker will hedge the floating rate payments by using, for example, forward rate agreements.
Computing the swap rate

- The requirement that the hedged swap have zero net PV is

$$\sum_{i=1}^{n} P(0, t_i) \left[ R - r_0(t_{i-1}, t_i) \right] = 0 \quad (8.1)$$

- Equation (8.1) can be rewritten as

$$R = \frac{\sum_{i=1}^{n} P(0, t_i) r_0(t_{i-1}, t_i)}{\sum_{i=1}^{n} P(0, t_i)} \quad (8.2)$$

where $\sum_{i=1}^{n} P(0, t_i) r_0(t_{i-1}, t_i)$ is the PV of interest payments implied by the strip of forward rates, and $\sum_{i=1}^{n} P(0, t_i)$ is the PV of a $1$ annuity when interest rates vary over time.
Computing the swap rate

- We can rewrite equation (8.2) to make it easier to interpret:

\[
R = \sum_{i=1}^{n} \left[ \frac{P(0, t_i)}{\sum_{j=1}^{n} P(0, t_j)} \right] r_0(t_{i-1}, t_i)
\]

Thus, the fixed swap rate is as a weighted average of the implied forward rates, where zero-coupon bond prices are used to determine the weights.
Computing the swap rate

• Alternative way to express the swap rate is

\[ R = \frac{1 - P(0, t_n)}{\sum_{i=1}^{n} P(0, t_i)} \]  \hspace{1cm} (8.3)

- using \( r_0(t_1,t_2) = \frac{P(0,t_1)}{P(0,t_2)} - 1 \)

This equation is equivalent to the formula for the coupon on a par coupon bond.

Thus, the swap rate is the coupon rate on a par coupon bond.

(firm that swaps floating for fixed ends up with economic equivalent of a fixed-rate bond)
The swap curve

• A set of swap rates at different maturities is called the *swap curve*.

• The swap curve should be consistent with the interest rate curve implied by the Eurodollar futures contract, which is used to hedge swaps.

• Recall that the Eurodollar futures contract provides a set of 3-month forward LIBOR rates. In turn, zero-coupon bond prices can be constructed from implied forward rates. Therefore, we can use this information to compute swap rates.
The swap curve

For example, the December swap rate can be computed using equation (8.3):

\[
\frac{(1 - 0.9485)}{(0.9830 + 0.9658 + 0.9485)} = 1.778\%.
\]

Multiplying 1.778\% by 4 to annualize the rate gives the December swap rate of 7.109\%.
The swap curve

- The swap spread is the difference between swap rates and Treasury-bond yields for comparable maturities.
The swap’s implicit loan balance

- Implicit borrowing and lending in a swap can be illustrated using the following graph, where the 10-year swap rate is 7.4667%:
The swap’s implicit loan balance

- In the above graph,
  
  ➢ Consider an investor who pays fixed and receives floating. This investor is paying a high rate in the early years of the swap, and hence is lending money. About halfway through the life of the swap, the Eurodollar forward rate exceeds the swap rate and the loan balance declines, falling to zero by the end of the swap.

  ➢ Therefore, the credit risk in this swap is borne, at least initially, by the fixed-rate payer, who is lending to the fixed-rate recipient.
Deferred swap

- A **deferred swap** is a swap that begins at some date in the future, but its swap rate is agreed upon today.

\[
R = \frac{\sum_{i=k}^{T} P(0, t_i) r_0(t_{i-1}, t_i)}{\sum_{i=k}^{T} P(0, t_i)}
\]

- The fixed rate on a deferred swap beginning in \( k \) periods is computed as

(8.4)
Why swap interest rates?

• Interest rate swaps permit firms to separate credit risk and interest rate risk.

➢ By swapping its interest rate exposure, a firm can pay the short-term interest rate it desires, while the long-term bondholders will continue to bear the credit risk.
Amortizing and accreting swaps

• An **amortizing swap** is a swap where the notional value is *declining* over time (e.g., floating rate mortgage).

• An **accreting swap** is a swap where the notional value is *growing* over time.

• The fixed swap rate is still a weighted average of implied forward rates, but now the weights also involve changing notional principle, $Q_t$:

$$R = \frac{\sum_{i=1}^{n} Q_t P(0,t_i) r(t_{i-1},t_i)}{\sum_{i=1}^{n} Q_t P(0,t_i)}$$

(8.7)
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