Lecture 12: Factor Pricing

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Overview

1. ICAPM – multiperiod economic model (last lecture)

2. Asset Pricing Theory (APT) – static statistical model
   - Merits of Factor Pricing
   - Exact Factor Pricing and Factor Pricing Errors
   - Factor Structure and Pricing Error Bounds
   - Single Factor and Beta Pricing (and CAPM)
   - (Factor) Mimicking Portfolios
   - Unobserved Factor Models
   - Multi-period outlook

3. Empirical Factor Pricing Models
   - Arbitrage Pricing Theory (APT) Factors
   - The Fama-French Factor Model + Momentum
The Merits of Factor Models

• Without any structure one has to estimate
  - J expected returns $E[R^j]$ (for each asset j)
  - J standard deviations
  - J(J-1)/2 co-variances

• Assume that the correlation between any two assets is explained by systematic components/factors, one can restrict attention to only K (non-diversifiable) factors
  - Advantages: Drastically reduces number of input variables
    - Models expected returns (priced risk)
    - Allows to estimate systematic risk
      (even if it is not priced, i.e. uncorrelated with SDF)
    - Analysts can specialize along factors
  - Drawbacks: Purely statistical model (no theory)
    (does not explain why factor deserves compensation: risk vs. mispricing)
    - Relies on past data and assumes stationarity
Factor Pricing Setup …

• $K$ factors $f_1, f_2, \ldots, f_K$
  - $\mathbb{E}[f_k] = 0$
  - $K$ is small relative to dimension of $\mathcal{M}$
  - $f_k$ are not necessarily in $\mathcal{M}$

• $\mathcal{F}$ space spanned by $f_1, \ldots, f_K, e$

• in payoffs

\[
x_j = \mathbb{E}(x_j)1 + \sum_{k=1}^{K} b_{jk} f_k + \delta_j,
\]

with $\delta_j \perp \mathcal{F}$, and in particular $\mathbb{E}[\delta_j] = 0$.

- $b_{j,k}$ factor loading of payoff $x_j$
...Factor Pricing Setup

• in returns

\[ r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j, \quad (1) \]

\[
\text{with } \beta_{jk} = \frac{b_{jk}}{p_j}, \text{ the factor loading of return } r_j, \\
\text{and } \epsilon_j = \frac{\delta_j}{p_j}.
\]

• Remarks:

➤ One can always choose orthogonal factors \( \text{Cov}[f_k, f_{k'}] = 0 \)
➤ Factors can be observable or unobservable
Factor Structure

• Definition of “factor structure:”

\[ r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j \] (1), where

\[ \text{cov}(\epsilon_j, \epsilon_i) = 0 \text{ if } i \neq j, \ E[\epsilon_j] = 0 \text{ and} \]
\[ \text{cov}(\epsilon_j, f_k) = 0 \text{ for each } (j, k). \]

• \( \Rightarrow \) risk can be split in *systematic* risk and *idiosyncratic (diversifiable)* risk
Exact vs. Approximate Factor Pricing

- Multiplying (1) by $k_q$ and taking expectations
  \[ 1 = E[r_j]E[k_q] + \sum_{k=1}^{K} \beta_{jk} E[k_q f_k] + E[k_q \epsilon_j]. \]
- Rearranging

\[
E[r_j] = \frac{1}{E[k_q]} + \sum_{k=1}^{K} \beta_{jk} \frac{-E[k_q f_k]}{E[k_q]} =: \gamma_k \quad \frac{E[k_q \epsilon_j]}{E[k_q]} =: \psi_j [\text{error}]
\]

- **Exact factor pricing:**
  - error: $\psi_j = 0$ (i.e. $\epsilon_j$ s orthogonal to $k_q$)
  - e.g. if $k_q \in \mathcal{F}$
Bound on Factor Pricing Error...

- Recall error: \( \psi_j = -\frac{E[k_q \epsilon_j]}{E[k_q]} \)
  - Note, if \( \exists \) risk-free asset and all \( f_k \in \mathcal{M} \), then \( \psi_j = -\bar{r}_q(\epsilon_j) \).
- If \( k_q \in \mathcal{F} \), then factor pricing is exact.
- If \( k_q \notin \mathcal{F} \), then \( k_q = k_q^\mathcal{F} + \eta \), with \( \eta \perp \mathcal{F} \), \( E[k_q \epsilon_j] = E[\eta \epsilon_j] \).
  - Let's make use of the Cauchy-Schwarz inequality (which holds for any two random variables \( z_1 \) and \( z_2 \))
    \[ |E[z_1 z_2]| \leq \sqrt{E[z_1^2]} \sqrt{E[z_2^2]} \]
    
    \[ |E[\eta \epsilon_j]| \leq \sqrt{E[\eta^2]} \sqrt{E[\epsilon_j^2]} = \sqrt{E[(k_q - k_q^\mathcal{F})^2]} \sigma(\epsilon_j) \].
  - Error-bound
    \[ |\psi_j| \leq \frac{1}{E[k_q]} \sigma(\epsilon_j) \sqrt{E[(k_q - k_q^\mathcal{F})^2]} \].
Error-Bound if Factor Structure Holds

• Factor structure \( \Rightarrow \) split idiosyncratic from systematic risk
• \( \Rightarrow \) all idiosyncratic risk \( \varepsilon_j \) are linearly independent and span space orthogonal to \( \mathcal{F} \). Hence, \( \eta = \sum_j a_j \varepsilon_j \)

• Note \( E[k_q \varepsilon_j] = E[\eta \varepsilon_j] = a_j E[\varepsilon_j^2] = a_j \sigma^2 (\varepsilon_j^2) \)

• Error \( \psi_j = - \frac{E[k_q \varepsilon_j]}{E[k_q]} = - \frac{1}{E[k_q]} a_j \sigma^2 (\varepsilon_j) \)

• Pythagorean Thm: If \( \{z_1, \ldots, z_n\} \) is orthogonal system in Hilbert space, then \( \| \sum_{i=1}^{n} z_i \|^2 = \sum_{i=1}^{n} \| z_i \|^2 \)
  ➢ Follows from def. of inner product and orthogonality
Error-Bound if Factor Structure Holds

Applying Pythagorean Thm to \( \eta = \sum_{j}^{J} a_j \epsilon_j \)

implies \( \sum_{j}^{J} a_j^2 \mathbb{E}[\epsilon_j^2] = ||\eta||^2 \)

\( \sum_{j}^{J} a_j^2 \sigma^2(\epsilon_j) = ||k_q - k_q^F||^2 \)

Multiply by \((1/\mathbb{E}[k_q]^2)\max_j\{\sigma^2(\epsilon_j)\}\) and making use of \( \sigma^2(\epsilon_j) \leq \max_j\{\sigma^2(\epsilon_j)\} \)

\[ \sum_{j}^{J} \psi_j^2 \leq \frac{1}{\mathbb{E}[k_q]^2} \mathbb{E}[(k_q - k_q^F)^2] \max_j\{\sigma^2(\epsilon_j)\} \]

RHS is constant for constant \( \max[\sigma^2(\epsilon_j)] \).

\( \Rightarrow \) For large J, most securities must have small pricing error

• Intuition for Approximate Factor Pricing:
  Idiosyncratic risk can be diversified away
One Factor Beta Model...

- Let $r$ be a risky frontier return and set
  \[ f = r - E[r] \] (i.e. $f$ has zero mean)
  - $q(f) = q(r) - q(E[r])$

- Risk free asset exists with gross return of $\tilde{r}$
  - $q(f) = 1 - E[r]/\tilde{r}$

- $f$ and $\tilde{r}$ span $\mathcal{E}$ and hence $k_q \in \mathcal{F}$

$\Rightarrow$ *Exact Factor Pricing*
One Factor Beta Model

Recall

\[ E[r_j] = \frac{1}{E[k_q]} + \sum_{k=1}^{K} \beta_{jk} \frac{-E[k_qf_k]}{E[k_q]} - \frac{E[k_q\epsilon_j]}{E[k_q]} =: \gamma_0 = \bar{r} \]

\[ =: \gamma_k = \psi_j = 0 \]

\[ E[r_j] = \bar{r} - \beta_j \bar{r} q(f) \]
\[ E[r_j] = \bar{r} - \beta_i \{E[r] - \bar{r}\} \]

Recall

\[ r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j \]

\[ \beta_j = \text{Cov}[r_j, f] / \text{Var}[f] = \text{Cov}[r_j, r] / \text{Var}[r] \]

If \( r_m \in \mathcal{E} \) then CAPM
Mimicking Portfolios...

- Regress on factor directly or on portfolio that mimics factor
  - Theoretical justification: project factor on $M$
  - Advantage: portfolios have smaller measurement error

- Suppose portfolio contains shares $\alpha_1, \ldots, \alpha_J$ with $\sum_j^J \alpha_j = 1$.

- Sensitivity of portfolio w.r.t. to factor $f_k$ is $\gamma_k = \sum_j \alpha_j \beta_{jk}$

- Idiosyncratic risk of portfolio is $\nu = \sum_j \alpha \varepsilon_j$
  - $\sigma^2(\nu) = \sum_j \alpha^2 \sigma(\varepsilon_j)$
  - diversification
Мимика портфелей

- Портфель чувствителен только к фактору \( k_0 \) (и идивидуальным рискам) если для каждого \( k \neq k_0 \) \( \gamma_k = \sum \alpha_j \beta_{jk} = 0 \), и \( \gamma_{k0} = \sum \alpha_j \beta_{jk0} \neq 0 \).

- Родименность пространства портфелей, чувствительных к определенному фактору, равна \( J-(K-1) \).

- Портфель копирует фактор \( k_0 \), если это портфель с наименьшим идивидуальным риском среди портфелей, чувствительных только к \( k_0 \).
Observable vs. Unobservable Factors...

- Observable factors: GDP, inflation etc.
- Unobservable factors:
  - Let data determine “abstract” factors
  - Mimic these factors with “mimicking portfolios”
  - Can always choose factors such that
    - factors are orthogonal, $\text{Cov}[f_k, f_{k'}] = 0$ for all $k \neq k'$
    - Factors satisfy “factor structure” (systemic & idiosyncratic risk)
    - Normalize variance of each factor to ONE
      $\Rightarrow$ pins down factor sensitivity (but not sign, - one can always change sign of factor)
...Unobservable Factors...

- Empirical content of factors
  - $\text{Cov}[r_i, r_j] = \sum_k \beta_{ik} \beta_{jk} \sigma^2(f_k)$
  - $\sigma^2(r_j) = \sum_k \beta_{jk} \beta_{jk} \sigma^2(f_k) + \sigma^2(\epsilon_j)$
  - $\sigma(f_k) = 1$ for $k = 1, \ldots, K$. (normalization)
  - In matrix notation
    - $\text{Cov}[r, r'] = \sum_k \beta_k' \beta_k \sigma^2(f_k) + D$,
      - where $\beta_k = (\beta_{1k}, \ldots, \beta_{Jk})$.
    - $\Omega = B B' + D$,
      - where $B_{jk} = \beta_{jk}$, and $D$ diagonal.
      - For PRINCIPAL COMPONENT ANALYSIS assume $D = 0$
        (if $D$ contains the same value along the diagonal it does affect eigenvalues but not eigenvectors – which we are after)
For any symmetric JxJ matrix A (like BB’), which is semi-positive definite, i.e. $y’Ay \geq 0$, there exist numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_J \geq 0$ and non-zero vectors $y_1, \ldots, y_J$ such that

- $y_j$ is an eigenvector of A assoc. w/ eigenvalue $\lambda_j$, that is $Ay_j = \lambda_j y_j$
- $\sum_j^J y_j^i y_j^i = 0$ for $j \neq j'$
- $\sum_j^J y_j^i y_j^i = 1$
- rank $(A) = $ number of non-zero $\lambda$ ‘s
- The $y_j$ ‘s are unique (except for sign) if the $\lambda_i$ ‘s are distinct

Let Y be the matrix with columns $(y_1, \ldots, y_J)$, and let $\Lambda$ the diagonal matrix with entries $\lambda_i$ then

$$ A = Y \sqrt{\Lambda} \sqrt{\Lambda} Y' $$
…Unobservable Factors

• If K-factor model is true, BB' is a symmetric positive semi-definite matrix of rank $K.$
  
  - Exactly K non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$ and associated eigenvectors $y_1, \ldots, y_K$
  
  - $Y_K$ the matrix with columns given by $y_1, \ldots, y_K$ \( \Lambda_K \) the diagonal matrix with entries $\lambda_i$, $j=1, \ldots, K.$

  $BB' = Y_K \sqrt{\Lambda_K} \sqrt{\Lambda_K} Y_K'$. 

  Hence,

  \[
  r_j = \sum_{k=1}^{K} (Y_K \sqrt{\Lambda_K})_{jk} f_k + \epsilon_j
  \]

• Factors are not identified but sensitivities are (except for sign.)

• In practice choose $K$ so that $\lambda_k$ is small for $k > K.$
Why more than ONE mimicking portfolio?

- Mimic (un)observable factors with portfolios
  [Projection of factor on asset span]

- Isn’t a *single* portfolio which mimics pricing kernel sufficient $\Rightarrow$ ONE factor

- So why multiple factors?
  - Not all assets are included (real estate, human capital …)
  - Other factors capture dynamic effects
    [since e.g. conditional $\neq$ unconditional. CAPM]
    (more later to this topic)
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APT Factors of Chen, Roll and Ross (1986)

1. Industrial production  
   (reflects changes in cash flow expectations)
2. Yield spread btw high risk and low risk corporate bonds  
   (reflects changes in risk preferences)
3. Difference between short- and long-term interest rate  
   (reflects shifts in time preferences)
4. Unanticipated inflation
5. Expected inflation (less important)

Note: The factors replicate market portfolio.
Fama-MacBeth 2 Stage Method

• **Stage 1:** Use *time series* data to obtain estimates for each individual stock’s $\beta^j$

  \[ R^j_t - R^f = \alpha + \beta^j (R^m_t - R^f_t) + \epsilon^j_t \]

  (e.g. use monthly data for last 5 years)

  Note: $\hat{\beta}^j$ is just an estimate [around true $\beta^j$]

• **Stage 2:** Use *cross sectional* data and estimated $\beta^j$s to estimate SML

  \[ R^j_{\text{next month}} = \alpha + b\hat{\beta}^j + e^j \]

  $b$=market risk premium
CAPM $\beta$–Testing Fama French (1992)

- Using newer data slope of SML $b$ is not significant (adding size and B/M)
- Dealing with econometrics problem:
  - $\hat{\beta}_j$ s are only noisy estimates, hence estimate of $b$ is biased
  - Solution:
    - Standard Answer: Find instrumental variable
    - Answer in Finance: Derive $\hat{\beta}$ estimates for portfolios
      - Group stocks in 10 x 10 groups sorted to size and estimated $\hat{\beta}_j$
      - Conduct Stage 1 of Fama-MacBeth for portfolios
      - Assign all stocks in same portfolio same $\beta$
      - Problem: Does not resolve insignificance
- CAPM predictions: $b$ is significant, all other variables insignificant
- Regressions: size and B/M are significant, $b$ becomes insignificant
  - Rejects CAPM
Book to Market and Size

Small “value“ companies have higher returns

AVERAGE RETURNS ON U.S. STOCKS DEPENDING ON SIZE AND B/M
Percent per month

High B/M is similar to low P/E, it means “value“. The opposite is “growth“.

Source: Mertens, Data from Fama and French (1992)
Fama French Three Factor Model

- Form 2x3 portfolios
  - Size factor (SMB)
    - Return of small minus big
  - Book/Market factor (HML)
    - Return of high minus low
- For \( R_t^{j} - R_t^{f} = \alpha^p + \beta^p (R_t^m - R_t^{f}) \)
  \( \alpha \)s are big and \( \beta \)s do not vary much
- For \( R_t^{p} - R_t^{f} = \alpha^p + \beta^p (R_t^m - R_t^{f}) + \gamma^p SMB_t + \delta^p HML_t \)
  (for each portfolio \( p \) using time series data)
  \( \alpha \)s are zero, coefficients significant, high \( R^2 \).
Fama French Three Factor Model

• Form 2x3 portfolios
  ➢ Size factor (SMB)
    • Return of small minus big
  ➢ Book/Market factor (HML)
    • Return of high minus low

• For \( R_t^j - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f) \)
  \( \alpha \)'s are big and \( \beta \)'s do not vary much

• For \( R_t^p - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f) + \gamma^p SMB_t^p + \delta^p HML_t^p \)
  (for each portfolio \( p \) using time series data)
  \( \alpha^p \)'s are zero, coefficients significant, high \( R^2 \).
Book to Market as a Predictor of Return

The chart illustrates the relationship between book-to-market ratios and annualized rates of return. It categorizes companies into two groups: High Book/Market and Low Book/Market. The annualized rate of return decreases as the book-to-market ratio increases, indicating that companies with a high book-to-market ratio tend to have lower returns and vice versa. This suggests that book-to-market can be a predictor of return, with Value stocks generally providing higher returns than Growth stocks.
Book to Market Equity of Portfolios Ranked by Beta

![Graph showing the relationship between book to market equity and beta. The x-axis represents beta ranging from 0.6 to 1.8, and the y-axis represents book to market equity ranging from 0.5 to 1.0. The data points suggest a positive correlation between beta and book to market equity.]
Adding Momentum Factor

• 5x5x5 portfolios

• Jegadeesh & Titman 1993 JF rank stocks according to performance to past 6 months
  ➢Momentum Factor
    Top Winner minus Bottom Losers Portfolios
Monthly Difference Between Winner and Loser Portfolios at Announcement Dates

Months Following 6 Month Performance Period
Cumulative Difference Between Winner and Loser Portfolios at Announcement Dates

Cumulative Difference Between Winner and Loser Portfolios at Announcement Dates

Months Following 6 Month Performance Period
Morgan Stanley’s Macro Proxy Model

• Factors
  ➢ GDP growth
  ➢ Long-term interest rates
  ➢ Foreign exchange (Yen, Euro, Pound basket)
  ➢ Market Factor
  ➢ Commodities or oil price index

• Factor-mimicking portfolios (“Macro Proxy”)
  ➢ Stage 1: Regress individual stocks on macro factors
  ➢ Stage 2: Create long-short portfolios of most and least sensitive stocks [5 quintiles]
    • Macro Proxy return predicts macro factor
Haugen’s view: The Evolution of Academic Finance

The Old Finance

1930's  40's  50's  60's  70's  80's  90's  beyond

Modern Finance

Modern Finance

Theme: Valuation Based on Rational Economic Behavior
Paradigms: Optimization Irrelevance CAPM EMH
(Markowitz) (Modigliani & Miller) (Sharpe, Lintner & Mossen) (Fama)
Foundation: Financial Economics
Haugen’s view: The Evolution of Academic Finance

The Old Finance

1930’s  40’s  50’s  60’s  70’s  80’s  90’s  beyond

Modern Finance

The New Finance

Theme: Inefficient Markets
Paradigms: Inductive *ad hoc* Factor Models
Expected Return Risk

Foundation: Statistics, Econometrics, and Psychology

Factor Pricing