Lecture 6: Competitive Equilibrium in the Growth Model (II)

ECO 503: Macroeconomic Theory I

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Fall 2014
Plan of Lecture

1. Sequence of markets CE
2. The growth model and the data
Sequence of Markets CE

- **Arrow-Debreu CE**
  - period 0: markets for *everything*

- **Sequence of Markets CE**: particular markets at particular points in time

<table>
<thead>
<tr>
<th>Period 0</th>
<th>Period 1</th>
<th>Period 2</th>
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<tbody>
<tr>
<td>market for period 0 capital,</td>
<td>...</td>
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<tr>
<td>period 0 labor,</td>
<td>...</td>
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<td>period 0 output,</td>
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<tr>
<td>period 0 labor,</td>
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<td>1 period ahead borrowing/leding</td>
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- Individ. formulates plan at $t = 0$, but executes it in real time
  - in contrast, in ADCE everything happens in period 0

- SOMCE features explicit **borrowing & lending**
  - riskless one-period bond that pays real interest rate $r_t$
Sequence of Market CE

**Definition**: A SOMCE for the growth model are sequences 
\( \{c_t, h_t, k_t, a_t, w_t, R_t, r_t\}_{t=0}^\infty \) s.t.

1. **(HH max)** Taking \( \{w_t, R_t, r_t\} \) as given, \( \{c_t, h_t, k_t, a_t\} \) solves

   \[
   \max_{\{c_t, h_t, k_t, a_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}
   \]

   \[
   c_t + k_{t+1} - (1 - \delta)k_t + a_{t+1} \leq R_t k_t + w_t h_t + (1 + r_t) a_t
   \]

   \[
   c_t \geq 0, \quad 0 \leq h_t \leq 1, \quad k_{t+1} \geq 0, \quad k_0 = \bar{k}_0, \quad a_0 = 0
   \]

   \[
   \lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1 + r_t} \right) a_{T+1} \geq 0 \quad (*)
   \]

2. **(Firm max)** Taking \( \{w_t, R_t, r_t\} \) as given, \( \{k_t, h_t\} \) solves

   \[
   \max_{k_t, h_t} F(k_t, h_t) - w_t h_t - R_t k_t \quad k_t \geq 0, \quad h_t \geq 0 \quad \forall t.
   \]

3. **(Market clearing)** For each \( t \):

   \[
   c_t + k_{t+1} - (1 - \delta)k_t = F(k_t, h_t)
   \]

   \[
   a_{t+1} = 0 \quad (**)
   \]
Comments

- \( a_t \) = HH bond holdings
  - \( a_t > 0 \): HH saves, \( a_t < 0 \): HH borrows
  - period-\( t \) price of bond that pays off at \( t + 1 \): \( q_t = 1/(1 + r_t) \)
  - some people like to write
    \[
    c_t + k_{t+1} - (1 - \delta)k_t + q_t b_{t+1} \leq R_t k_t + w_t h_t + b_t
    \]
  - this is equivalent with \( b_t = (1 + r_t) a_t \) and \( q_t = 1/(1 + r_t) \)

- Interpretation of bond market clearing condition (***)
  - bonds are in zero net supply
  - more generally, in economy with individuals \( i = 1, \ldots, N \)
    \[
    \sum_{i=1}^{N} a_{i,t+1} = 0
    \]
  - for every dollar borrowed, someone else saves a dollar
  - here only one type, so \( a_{t+1} = 0 \).
  - Q: since \( a_t = 0 \), why not eliminate? A: need to know eq. \( r_t \)
• \((\ast)\) is a so-called “no-Ponzi condition”
  • with period budget constraints only, individuals could choose time paths with \(a_t \to -\infty\)
  • no-Ponzi condition \((\ast)\) rules out such time paths: \(a_t\) cannot become too negative
  • implies that sequence of budget constraints can be written as present-value (or time-zero) budget constraint
  • return to this momentarily

• Could have written firm’s problem as

\[
\max_{\{k_t, h_t\}} \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (F(k_t, h_t) - w_t h_t - R_t k_t) \quad k_t \geq 0, \quad h_t \geq 0
\]

but this is a sequence of static problems so can split them up
Sequence BC + no-Ponzi $\Rightarrow$ PVBC

- **Result:** If $\{c_t, i_t, h_t\}$ satisfy the sequence budget constraint

$$c_t + i_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t) a_t$$

and if the no-Ponzi condition (*) holds with equality, then $\{c_t, i_t, h_t\}$ satisfy the present value budget constraint

$$\sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (c_t + i_t) = \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (R_t k_t + w_t h_t)$$

- **Proof:** next slide
Proof

• Write period $t$ budget constraint as

$$
\frac{1}{1 + r_t} a_{t+1} = \frac{1}{1 + r_t} (R_t k_t + w_t h_t - c_t - i_t) + a_t
$$

• At $t = 0, t = 1, ...$

$$
\frac{1}{1 + r_0} a_1 = \frac{1}{1 + r_0} (R_0 k_0 + w_0 h_0 - c_0 - i_0) + a_0
$$

$$
\frac{1}{1 + r_0} \frac{1}{1 + r_1} a_2 = \frac{1}{1 + r_0} \frac{1}{1 + r_1} (R_1 k_1 + w_1 h_1 - c_1 - i_1)
+ \frac{1}{1 + r_0} (R_0 k_0 + w_0 h_0 - c_0 - i_0) + a_0
$$

• By induction/repeated substitution

$$
\left(\prod_{t=0}^{T} \frac{1}{1 + r_t}\right) a_{T+1} = \sum_{t=0}^{T} \left(\prod_{s=0}^{t} \frac{1}{1 + r_s}\right) (R_t k_t + w_t h_t - i_t - c_t)
$$

• Result follows from taking $T \rightarrow \infty$ and imposing $(\star)$
Why no-Ponzi Condition?

- Expression also provides some intuition for no-Ponzi condition

\[
\left( \prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = \sum_{t=0}^{T} \left( \prod_{s=0}^{t} \frac{1}{1+r_s} \right) (R_t k_t + w_t h_t - i_t - c_t)
\]

- Suppose for the moment this were a finite horizon economy
  - would impose: die without debt, i.e. \( a_{T+1} \geq 0 \)
  - in fact, HH’s would always choose \( a_{T+1} = 0 \)

- Right analogue for infinite horizon economy

\[
\lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0
\]

and HH’s choose \( \{a_t\} \) so that this holds with equality

- no-Ponzi condition **not needed** for physical capital because natural constraint \( k_t \geq 0 \).
Characterizing SOMCE

- Necessary conditions for consumer problem \((h_t = 1 \text{ wlog})\)
  
  \[
  c_t : \quad \beta^t u'(c_t) = \lambda_t = \text{multiplier on period } t \text{ b.c.} \quad (1)
  \]
  
  \[
  k_{t+1} : \quad \lambda_t = \lambda_{t+1}(R_{t+1} + 1 - \delta) \quad (2)
  \]
  
  \[
  a_{t+1} : \quad \lambda_t = \lambda_{t+1}(1 + r_{t+1}) \quad (3)
  \]
  
  \[
  c_t + k_{t+1} - (1 - \delta)k_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t)a_t \quad (4)
  \]
  
  no-Ponzi: \(\lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0 \quad (5)\)
  
  TVC on \(k\): \(\lim_{T \to \infty} \beta^T u'(c_T)k_{T+1} = 0\) \(\quad (6)\)
  
  TVC on \(a\): \(\lim_{T \to \infty} \beta^T u'(c_T)a_{T+1} = 0\) \(\quad (7)\)
  
  initial: \(k_0 = \bar{k}_0, \quad a_0 = 0\) \(\quad (8)\)
Characterizing SOMCE

• Necessary conditions for **firm problem**

\[ F_k(k_t, h_t) = R_t, \quad F_h(k_t, h_t) = \omega_t \tag{9} \]

• **Market clearing**

\[ c_t + k_{t+1} - (1 - \delta)k_t = F(k_t, h_t), \quad a_{t+1} = 0 \tag{10} \]
Characterizing SOMCE

• (1), (3) and (5)

\[ \beta^T u'(c_T) = \lambda_T = \prod_{t=0}^T \frac{1}{1 + r_t} \]

\[ \Rightarrow \lim_{T \to \infty} \beta^T u'(c_T) a_{T+1} \geq 0 \]

• No-Ponzi condition looks very similar to TVC on \( \{a_t\} \)

• But no-Ponzi and TVC are **different conditions**

• Kamihigashi (2008) “A no-Ponzi-game condition is a constraint that prevents overaccumulation of debt, while a typical transversality condition is an optimality condition that rules out overaccumulation of wealth. They place opposite restrictions, and should not be confused.”
Characterizing SOMCE

- (2) and (3)
  \[ 1 + r_{t+1} = R_{t+1} + 1 - \delta \]
  i.e. rate of return on bonds = rate of return on capital
  - arbitrage condition
  - if this holds, HH is indifferent between a and k

- (1), (2) and (9) ⇒
  \[
  \frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{11}
  \]

- (11) + TVC (6) + initial condition (8) + market clearing (10)
  = same set of equations as for SP problem

- Hence: SOMCE allocation is same as social planner’s allocation
  - this is actually somewhat surprising, see next slide
Why is SOMCE allocation $= SP$’s alloc.?

• Relative to ADCE, we closed down many markets

• Q: Why do we still get SP solution even though we closed down many markets?

• A: We only closed down markets that didn’t matter

• In fact, ADCE and SOMCE are equivalent
Equivalence of SOMCE and ADCE

- Recall HH’s problem in ADCE (last lecture):

\[
\max_{\{c_t, h_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \\
\sum_{t=0}^{\infty} p_t (c_t + k_{t+1} - (1 - \delta)k_t) \leq \sum_{t=0}^{\infty} p_t (R_t k_t + w_t h_t)
\]

- Have shown earlier: HH’s problem in SOMCE is same with present-value budget constraint

\[
\sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1 + r_s} \right) (R_t k_t + w_t h_t)
\]

- Clearly these are equivalent
  - ADCE is SOMCE with \( p_t = \prod_{s=0}^{t} \frac{1}{1 + r_s} \)
  - SOMCE is ADCE with \( 1 + r_{t+1} = p_t / p_{t+1} \)

- Firm’s problems are also equivalent.
Why is SOMCE allocation = SP’s alloc.?

- riskless one-period bond is surprisingly powerful
- one period ahead borrowing and lending ⇒ arbitrary period ahead borrowing and lending
- When is SOMCE allocation with one-period bonds ≠ SP’s allocation? That is, when do the welfare theorems fail?
  - risk (idiosyncratic or aggregate)
    - welfare theorems may hold if sufficiently rich insurance markets
  - “financial frictions.” Examples:
    - interest rate = \( r_t(a_t) \) with \( r_t' \neq 0 \).
    - in more general environments: borrowing constraint \( -a_t \leq 0 \) or collateral constraints (need to back debt with collateral)
      \[ -a_{t+1} \leq \theta k_{t+1} \]
  - ...
