Lectures 8: Policy Analysis in the Growth Model
(Capital Taxation)

ECO 503: Macroeconomic Theory I

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Policy Analysis in the Growth Model

- Classic question: what are the consequences for allocations and welfare of policy $x$?
- Today: $x$ = capital income taxation
- but approach works more generally
Capital Taxes in the U.S.

- U.S. top marginal tax rates (from Saez, Slemrod and Giertz, 2012, Table A1)
Capital Taxation in Theory

- Most influential: Chamley and Judd’s zero capital tax result
  - somewhat more precisely: in the long-run, the optimal linear capital income tax should be zero
  - perhaps even reflected in observed policy (see previous slide)
Plan

1. Capital income taxation and redistribution
   - a growth model with capitalists and workers
   - “Ramsey taxation” (Judd, 1985)
   - critique by Straub and Werning (2014)

2. Capital income taxation without redistribution
   - “Ramsey taxation” (Chamley, 1986)
   - only quick overview

3. Summary: takeaway on capital taxation
Growth Model with Capitalists & Workers

- Consider a variant of the growth model with two types of individuals:
  - **capitalists**: rep. capitalist derives all income from returns to capital
  - **workers**: rep. worker derives all income from labor income

- Originally due to Judd (1985), use discrete-time formulation from Straub and Werning (2014)

- Two reasons why variant is better model for thinking about capital income taxation than standard growth model
  - some distributional conflict (as opposed to rep. agent)
  - math turns out to be easier

- End of lecture: capital taxation in representative agent model (Chamley, 1986)
Growth Model with Capitalists & Workers

• **Preferences**
  
  • capitalist
    \[
    \sum_{t=0}^{\infty} \beta^t U(C_t), \quad U(C) = \frac{C^{1-\sigma}}{1 - \sigma}
    \]
  
  • workers
    \[
    \sum_{t=0}^{\infty} \beta^t u(c_t)
    \]

• **Technology**

  \[
  c_t + C_t + k_{t+1} = F(k_t, h_t) + (1 - \delta)k_t
  \]

• **Endowments**: capitalists own \( k_0 = \hat{k}_0 \) units of capital
Competitive Equilibrium without Taxes

- **Definition**: A SOMCE for the growth model with capitalists and workers are sequences \( \{c_t, h_t, k_t, a_t, w_t, r_t\}_{t=0}^{\infty} \) s.t.

1. (Capitalist max) Taking \( \{r_t\} \) as given, \( \{C_t, a_t\} \) solves

\[
\max_{\{C_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.}
\]

\[
C_t + a_{t+1} = (1 + r_t)a_t, \quad \lim_{T \to \infty} \left( \prod_{s=0}^{T} \frac{1}{1+r_s} \right) a_{T+1} \geq 0, \quad a_0 = \hat{k}_0.
\]

2. (Worker max) Taking \( \{w_t\} \) as given, \( \{c_t, h_t\} \) solves

\[
\max_{\{c_t, h_{t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad c_t = w_t h_t
\]

3. (Firm max) Taking \( \{w_t, r_t\} \) as given \( \{k_t, h_t\} \) solves

\[
\max_{\{k_t, h_{t}\}} \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1+r_s} \right) (F(k_t, h_t) - w_t h_t - i_t), \quad k_{t+1} = i_t + (1-\delta)k_t
\]

4. (Market clearing) For each \( t \):

\[
c_t + C_t + k_{t+1} = F(k_t, h_t) + (1-\delta)k_t, \quad a_t = k_t
\]
Comments

- Only capitalist can save
- Worker cannot save, lives “hand to mouth”
- Work with decentralization in which
  - firms own capital
  - capitalists save in riskless bond
  - in contrast, in last lecture: households owned capital, rented it to firms
- Relative to Straub and Werning
  - make notation as similar as possible to last lecture
  - impose no-Ponzi condition rather than borrowing limit $a_{t+1} \geq 0$ (doesn’t matter)
Necessary Conditions

- **Necessary conditions for** capitalist **problem**

  \[ U'(C_t) = \beta (1 + r_{t+1}) U'(C_{t+1}) \]  
  \[ 0 = \lim_{T \to \infty} \beta^T U'(C_T) a_{T+1} \]  

- **Solution to** worker **problem**

  \[ h_t = 1, \quad c_t = w_t \]

- **Necessary conditions for** firm **problem**

  \[ F_h(k_t, h_t) = w_t \]
  \[ F_k(k_t, h_t) + 1 - \delta = 1 + r_t \]  

- **Market Clearing**

  \[ c_t + C_t + k_{t+1} = F(k_t, h_t) + (1 - \delta) k_t \]
Necessary Conditions

• (6) is same no-arbitrage condition we had in last lecture, but now coming directly from firm’s problem

• Combining (1) and (6) and defining $F(k_t, 1) = f(k_t)$ we get

$$U'(C_t) = \beta U'(C_t)(f'(k_{t+1}) + 1 - \delta)$$

• Same condition as usual, except that $C_t$ is consumption of capitalists

• In steady state $C_t = C^*, c_t = c^*, k_t = k^*$

$$f'(k^*) + 1 - \delta = \frac{1}{\beta}$$

⇒ same steady state as standard growth model.
Analytic Solution in Special Case: $\sigma = 1$

- **Lemma**: with $\sigma = 1$ capitalists save a constant fraction $\beta$
  
  \[ a_{t+1} = \beta (1 + r_t) a_t, \quad C_t = (1 - \beta)(1 + r_t) a_t \]

- **Proof**: “guess and verify”. Consider nec. cond’s w/ $\sigma = 1$

  \[ \frac{C_{t+1}}{C_t} = \beta (1 + r_{t+1}) \quad (\ast) \]

  \[ 0 = \lim_{T \to \infty} \beta^T \frac{a_{T+1}}{C_T} \]

  \[ C_t + a_{t+1} = R_t a_t \]

- Guess $C_t = (1 - s)(1 + r_t) a_t$. From (\ast)

  \[ \frac{(1 - s)(1 + r_{t+1}) a_{t+1}}{(1 - s)(1 + r_t) a_t} = \beta (1 + r_{t+1}) \quad \Rightarrow \quad \frac{a_{t+1}}{a_t} = \beta (1 + r_t) \]

  i.e. $s = \beta. \square$
\[ \sigma = 1: \text{Intuition for Constant Saving Rate} \]

- Log utility \( \Rightarrow \) offsetting income and substitution effects
  - \((a_{t+1}, C_t)\) do **not** depend on \( r_{t+1} \)
- \(1/\sigma = \text{"intertemporal elasticity of substitution (IES)"} \)
  - low \( \sigma \Rightarrow U \text{ close to linear ...} \)
  - ... capitalists like to substitute intertemporally ("high IES")
- To understand, consider effect of unexpected increase of \( r_{t+1} \)
  - \( \sigma > 1: \text{income effect dominates } \Rightarrow C_t \uparrow, a_{t+1} \downarrow \)
  - \( \sigma < 1: \text{substitution effect dominates } \Rightarrow C_t \downarrow, a_{t+1} \uparrow \)
  - \( \sigma = 1: \text{income \\& subst. effects cancel } \Rightarrow C_t, a_{t+1} \text{ constant} \)
- Same logic as in Lecture 4
  - there condition was \( \sigma \geq \alpha \) where \( \alpha = \text{curvature of prod. fn.} \)
  - reason for difference: planner in Lecture 4 faced concave saving technology, \( \varepsilon k_t^\alpha \)
  - ... here instead, capitalists face linear saving technology \(((1 + r_t)a_t)\). In effect, \( \alpha = 1. \)
Analytic Solution in Special Case: $\sigma = 1$

- Necessary conditions reduce to
  
  \[ k_{t+1} = \beta(f'(k_t) + 1 - \delta)k_t \]  
  
  \[ C_t = (1 - \beta)(f'(k_t) + 1 - \delta)k_t \]  
  
  \[ c_t = f(k_t) - f'(k_t)k_t \]

  (used $F = F_k k + F_h h$ and so $F_h(k_t, 1) = f(k_t) - f'(k_t)k_t$)

- Model basically boils down to **Solow model**
  
  - e.g. with $f(k) = Ak^\alpha$
    
    \[ k_{t+1} = \alpha\beta Ak_t^\alpha + \beta(1 - \delta)k_t \]
  
  - effective saving rate $\alpha\beta$ and depreciation term $\beta(1 - \delta)$

- Extremely convenient: compute entire transition by hand
  
  - no need for phase diagram etc, simply do Solow zig-zag graph
  
  - but still same steady state at standard growth model
    
    \[ f'(k^*) = 1/\beta + 1 - \delta \]
Policy in GE Models

• Next: policy in growth model with capitalists and workers

• Questions about policy need to be **well posed**
  • example of question that is not well-posed: “What happens if we introduce a proportional tax $\tau$ on capital?”
  • reason: if a policy raises revenue (or requires expenditure), then one must specify what is done with the revenue (where the revenue comes from)

• There are many possible uses of revenue $\Rightarrow$ many possible exercises

• Here, ask: What are the consequences of introducing
  • a proportional (linear) tax on capital income of $\tau_t$ when the revenues are used to fund
    • constant government consumption $g \geq 0$ and
    • a lump-sum transfer to workers $T_t$ with period-by-period budget balance?
Competitive Equilibrium with Taxes

**Definition:** A SOMCE with taxes for the growth model with capitalists and workers are sequences
\[
\{c_t, h_t, k_t, a_t, w_t, r_t, \tau_t, T_t\}_{t=0}^{\infty}
\]
s.t.

1. (Capitalist max) Taking \(\{r_t, \tau_t\}\) as given, \(\{C_t, a_t\}\) solves
   \[
   \max_{\{C_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.}
   \]
   \[
   C_t + a_{t+1} = (1 - \tau_t)(1 + r_t)a_t, \quad \lim_{T \to \infty} \left( \prod_{s=0}^{T} \frac{1}{1+r_s} \right) a_{T+1} \geq 0, a_0 = \hat{k}_0.
   \]

2. (Worker max) Taking \(\{w_t\}\) as given, \(\{c_t, h_t\}\) solves
   \[
   \max_{\{c_t, h_{t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}
   \]
   \[
   c_t = w_t h_t + T_t
   \]

3. (Firm max) Taking \(\{w_t, r_t\}\) as given \(\{k_t, h_t\}\) solves
   \[
   \max_{\{k_t, h_{t}\}} \sum_{t=0}^{\infty} \left( \prod_{s=0}^{t} \frac{1}{1+r_s} \right) (F(k_t, h_t) - w_t h_t - i_t), \quad k_{t+1} = i_t + (1-\delta)k_t
   \]
Competitive Equilibrium with Taxes

- **Definition:** A SOMCE with taxes for the growth model with capitalists and workers are sequences \(\{c_t, h_t, k_t, a_t, w_t, r_t, \tau_t T_t\}_{t=0}^{\infty}\) s.t.

  4. (Government) For each \(t\)

  \[g + T_t = \tau_t k_t\]

  5. (Market clearing) For each \(t\):

  \[c_t + C_t + k_{t+1} = F(k_t, h_t) + (1 - \delta)k_t, \quad a_t = k_t\]
Tax is **linear** as opposed to **non-linear** tax function \( \tilde{\tau} \)

\[
C_t + a_{t+1} = (1 + r_t) a_t - \tilde{\tau}((1 + r_t) a_t)
\]

with \( \tilde{\tau}'' \neq 0 \) (e.g. \( \tilde{\tau}'' > 0 \) = progressive)
Characterizing CE with Taxes

• Necessary conditions unchanged except for
  \[ U'(C_t) = \beta(1 - \tau_{t+1})(1 + r_{t+1})U'(C_{t+1}) \]
  and resource constraint
• Therefore
  \[ U'(C_t) = \beta U'(C_{t+1})(1 - \tau_{t+1})(f'(k_{t+1}) + 1 - \delta) \]
• For any \( \{\tau_t\}_{t=0}^{\infty} \) can use shooting algorithm to solve for eqm
  • natural initial condition: steady state without taxes
• What about steady state with taxes? Suppose \( \tau_t = \tau \). Then
  \[ (1 - \tau)(f'(k^*) + 1 - \delta) = \frac{1}{\beta} \]
  Hence higher \( \tau \uparrow \Rightarrow k^* \downarrow \), e.g. if \( f(k) = Ak^\alpha \)
  \[ k^* = \left( \frac{\alpha A}{\frac{1}{\beta(1-\tau)} + 1 - \delta} \right)^{\frac{1}{1-\alpha}} \]
Ramsey Taxation

• So far: **positive** analysis
  • what is the effect of $\tau_t$ ...?

• Now: **normative**
  • what is the **optimal** $\tau_t$

• **Ramsey problem**: find $\{\tau_t\}$ that produces a CE with taxes with highest utility for agents (capitalists and workers).

• that is, find optimal $\{\tau_t\}$ subject to the fact that agents behave competitively for those taxes

• Important assumption
Ramsey Problem

- Need to take stand on objective of policy
- Here use
  \[ \sum_{t=0}^{\infty} \beta^t (u(c_t) + \gamma U(C_t)) \]
  for a “Pareto weight” \( \gamma \geq 0 \)
  - \( \gamma = 0 \): only care about workers
  - \( \gamma \to \infty \): only care about capitalists
Ramsey Problem

- Recall necessary conditions for CE with taxes

\[ U'(C_t) = \beta(1 + r_{t+1})(1 - \tau_{t+1})U'(C_{t+1}) \]  
\[ 0 = \lim_{T \to \infty} \beta^T U'(C_T)a_{T+1} \]  
\[ C_t + a_{t+1} = (1 - \tau_t)(1 + r_t)a_t \]  
\[ c_t = w_t + T_t \]  
\[ F_h(k_t, 1) = w_t \]  
\[ F_k(k_t, 1) + 1 - \delta = 1 + r_t \]  
\[ c_t + C_t + g + k_{t+1} = F(k_t, 1) + (1 - \delta)k_t \]  
\[ k_t = a_t \]  
\[ a_0 = k_0 = \hat{k}_0 \]

- Ramsey problem is

\[
\max_{\{\tau_t, c_t, C_t, k_{t+1}, a_{t+1}, w_t, r_t\}} \sum_{t=0}^{\infty} \beta^t (u(c_t) + \gamma U(C_t)) \quad \text{s.t.} \quad (1)-(9)
\]
Ramsey Problem

- Can simplify by combining/eliminating some of the constraints

- From (3) and (8)

\[(1 - \tau_t)(1 + r_t) = \frac{C_t}{k_t} + \frac{k_{t+1}}{k_t}\]

- Substituting into (1)

\[U'(C_{t-1})k_t = \beta U'(C_t)(C_t + k_{t+1})\]

- Write \(F(k_t, 1) = f(k_t)\) as usual

- Walras’ Law: can drop one budget constraint or resource constraint. Drop (4).

- Also drop (5) and (6) since \(\{r_t, w_t\}_{t=0}^{\infty}\) only show up in equations we already dropped.
After simplifications:

\[
\max_{\{c_t, C_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t (u(c_t) + \gamma U(C_t)) \quad \text{s.t.}
\]

\[
c_t + C_t + g + k_{t+1} = f(k_t) + (1 - \delta)k_t
\]

\[
\beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t
\]

\[
\lim_{T \to \infty} \beta^T U'(C_T)k_{T+1} = 0
\]
Comments

• Note: problem only in terms of allocation

• Given optimal \( \{c_t, C_t, k_{t+1}\}_{t=0}^{\infty} \), can always back out taxes and prices

\[
\begin{align*}
  w_t &= F_h(k_t, 1) = f(k_t) - f'(k_t)k_t \\
r_t &= F_k(k_t, 1) - \delta = f'(k_t) - \delta \\
1 - \tau_t &= \frac{1}{f'(k_t) + 1 - \delta} \frac{U'(C_t)}{\beta U'(C_{t+1})}
\end{align*}
\]

• In other applications, typically combine constraints in different way, leading to so-called “implementability” condition.
  • same outcome: Ramsey problem in terms of allocations only

• But here follow Judd (1985) and Straub and Werning (2014). Easier to work with.
First order conditions

- **Lagrangean**

\[
\mathcal{L} = \sum_{t=0}^{\infty} \left\{ \beta^t (u(c_t) + \gamma U(C_t)) + \beta^t \lambda_t (f(k_t) + (1 - \delta)k_t - c_t - C_t - g - k_{t+1}) + \beta^t \mu_t (\beta U'(C_t)(C_t + k_{t+1}) - U'(C_{t-1})k_t) \right\}
\]

- **First order conditions (use that \( U'(C_t)C_t = C_t^{1-\sigma} \))**

\[
c_t : \quad 0 = u'(c_t) - \lambda_t \tag{1}
\]
\[
C_t : \quad 0 = \gamma U'(C_t) - \lambda_t - \beta \mu_{t+1} U''(C_t)k_{t+1} + \beta \mu_t ((1 - \sigma) U'(C_t) + U''(C_t)k_{t+1}) \tag{2}
\]
\[
k_{t+1} : \quad 0 = -\lambda_t + \mu_t \beta U'(C_t) + \beta \lambda_{t+1} (f'(k_{t+1}) + 1 - \delta) - \beta \mu_{t+1} U'(C_t) \tag{3}
\]
Tricky Detail: \( C_{-1} \)

- Treated \( C_t \) as a state variable, even though it’s a jump var
  - \( C_{-1} \) is not-predetermined

- Can show: multiplier \( \mu_t \) corresponding to \( \{C_t\} \) has to satisfy
  \[
  \mu_0 = 0
  \]

- Heuristic derivation: for any \((k_0, C_{-1})\) define \( V(k_0, C_{-1}) \) by

  \[
  V(k_0, C_{-1}) = \max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t(u(c_t) + \gamma U(C_t)) \quad \text{s.t.}
  \]

  \[
  c_t + C_t + g + k_{t+1} = f(k_t) + (1 - \delta)k_t
  \]
  \[
  \beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t
  \]
  \[
  \lim_{T \to \infty} \beta^T U'(C_T)k_{T+1} = 0
  \]

- \( C_{-1} \) pinned down from \( V_C(k_0, C_{-1}) = 0 \). Envelope condition

  \[
  V_C(k_0, C_{-1}) = \frac{\partial L}{\partial C_{-1}} = -\mu_0 U''(C_{-1})k_0 \quad \Rightarrow \quad \mu_0 = 0
  \]
First order conditions

• Manipulate (2) as follows

\[-\beta \mu_{t+1} U''(C_t) k_{t+1} = -\gamma U'(C_t) + \lambda_t - \beta \mu_t ( (1-\sigma) U'(C_t) + U''(C_t) k_{t+1} ) \]

Use that \( U''(C_t) k_{t+1} = -\sigma U'(C_t) \kappa_{t+1}, \kappa_{t+1} = k_{t+1}/C_t \)

\[\mu_{t+1} \beta \sigma U'(C_t) \kappa_{t+1} = \beta \mu_t ( (\sigma-1) U'(C_t) + U'(C_t) \kappa_{t+1} - \gamma U'(C_t) + \lambda_t \]

\[\mu_{t+1} = \mu_t \left( \frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{\lambda_t / U'(C_t) - \gamma}{\beta \sigma \kappa_{t+1}} \]

\[\mu_{t+1} = \mu_t \left( \frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1 - \gamma v_t}{\beta \sigma \kappa_{t+1} v_t}, \quad v_t = \frac{U'(C_t)}{u'(c_t)} \]

• Manipulate (3) as follows

\[\beta \lambda_{t+1} (f'(k_{t+1}) + 1 - \delta) = \lambda_t - \mu_t \beta U'(C_t) + \beta \mu_{t+1} U'(C_t) \]

Dividing by \( \beta \lambda_t \) and using \( \lambda_t = u'(c_t), v_t = U'(C_t)/u'(c_t) \)

\[\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + v_t (\mu_{t+1} - \mu_t) \quad (4)\]
First order conditions

- Using these manipulations we obtain

\[ \mu_0 = 0 \] (1)
\[ u'(c_t) = \lambda_t \] (2)
\[ \mu_{t+1} = \mu_t \left( \frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} v_t} (1 - \gamma v_t) \] (3)
\[ \frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + v_t (\mu_{t+1} - \mu_t) \] (4)

where \( \kappa_t = k_t / C_{t-1}, \ v_t = U'(C_t) / u'(c_t) \)

- Straub and Werning find it convenient to denote (note \( R_t \neq \) rental rate)

\[ R_t^e = f'(k_t) + 1 - \delta \]
\[ R_t = (1 - \tau_t)(f'(k_t) + 1 - \delta) = \frac{U'(C_t)}{\beta U'(C_{t+1})} \] (5)
\[ \tau = 0 \iff R_t^e / R_t = 1 \]
First order conditions

Theorem (Judd, 1985)

Suppose quantities and multipliers converge to an interior steady state, i.e. $c_t, C_t, k_{t+1}$ converge to positive values, and $\mu_t$ converges. Then the tax on capital is zero in the limit: $R_t^e / R_t \to 1$.

• **Proof:** Theorem assumes $(c_t, C_t, k_t, \mu_t) \to (c^*, C^*, k^*, \mu^*)$. Hence also $(v_t, \kappa_t) \to (v^*, \kappa^*)$.

• From (4) with $c_t = c_{t+1} = c^*$

\[ R_t^e \to R_{t+1}^e = \frac{1}{\beta}, \]

• Similarly, from (5) with $C_t^* = C_{t+1}^* = C^*$

\[ R_t \to R^* = \frac{1}{\beta}, \]

• Hence $R_t^*/R_t \to 1$ or equivalently $\tau_t \to 0.\blacksquare$
• **Theorem** seems to prove: capital taxes converge to zero in the long-run

• Really striking: this is true **even if** \( \gamma = 0 \), i.e. Ramsey planner only cares about workers!

• Is this really true? Let’s consider again the tractable case with log utility, \( \sigma = 1 \)
Ramsey Problem for $\sigma = 1, \gamma = 0$

- Recall analytic solution for capitalists’s saving decision
  
  $$a_{t+1} = s(1 - \tau_t)(1 + r_t)a_t, \quad C_t = (1 - s)(1 - \tau_t)(1 + r_t)a_t$$

  with $s = \beta$. Follow Straub-Werning in writing $s$, could come from somewhere else than $\sigma = 1$ assumption.

- Using $C_t = \frac{1-s}{s}k_{t+1}$, resource constraint becomes
  
  $$c_t + \frac{1}{s}k_{t+1} + g = f(k_{t+1}) + (1 - \delta)k_t$$

- Also assume $\gamma = 0$ (planner only cares about workers).

- Ramsey problem with $\sigma = 1, \gamma = 0$:
  
  $$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

  $$c_t + \frac{1}{s}k_{t+1} + g = f(k_{t+1}) + (1 - \delta)k_t$$

- Mathematically equivalent to standard growth model.
Ramsey Problem for $\sigma = 1, \gamma = 0$

- Euler equation is

$$u'(c_t) = s\beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) \quad (*)$$

- Because this is equivalent to growth model
  - unique interior steady state
  $$1 = s\beta(f'(k^*) + 1 - \delta)$$
  - globally stable
- With $R^* = 1/s$ and $R^{e*} = f'(k^*) + 1 - \delta$ have
  $$\frac{R^e}{R} = \frac{1}{\beta} \quad \Rightarrow \quad \tau^* = 1 - \beta > 0$$

- **Counterexample** to zero long-run capital taxes.
What Went Wrong?

- Crucial part of Judd’s Theorem: “**Suppose** quantities and multipliers converge to an interior steady state ...”
- Turns out this doesn’t happen: **multipliers explode**!
- Consider planner’s equations (3), (4) in case $\sigma = 1, \gamma = 0$

\[
\mu_{t+1} = \mu_t + \frac{1}{\beta \kappa_{t+1} v_t} \quad (3')
\]

\[
\frac{u'(c_{t+1})}{u'(c_t)} (f'(k_{t+1}) + 1 - \delta) = \frac{1}{\beta} + v_t (\mu_{t+1} - \mu_t) \quad (4')
\]

- Judd: **if** $\mu_t \to \mu^*$, then $\tau_t \to 0$ (follows from (4'))
- But from (3') $\mu_{t+1} > \mu_t$ for all $t \Rightarrow \mu_t \to \infty$
- In fact, with log-utility

\[
\kappa_{t+1} = \frac{k_{t+1}}{C_t} = \frac{s}{1 - s} \Rightarrow v_t (\mu_{t+1} - \mu_t) = \frac{1}{\beta \kappa_{t+1}} = \frac{1 - s}{\beta s}
\]

and so (4) implies (*) on previous slide and $\tau^* = 1 - \beta$
General Case $\sigma \neq 1$

- Straub and Werning (2014) analyze general case
- Not surprisingly, asymptotic behavior of $\tau_t$ different whether
  - $\sigma > 1$: positive limit tax
  - $\sigma < 1$: zero limit tax
- This is where the meat of the paper is
Proposition

If \( \sigma > 1 \) and \( \gamma = 0 \) then for any initial \( k_0 \) the solution to the planning problem converges to \( c_t \to 0, k_t \to k_g, C_t \to \frac{1-\beta}{\beta} k_g \), with a positive limit tax on wealth: \( 1 - \frac{R_t}{R^*_t} \to \tau_g > 0 \). The limit tax is decreasing in spending \( g \), with \( \tau_g \to 1 \) as \( g \to 0 \).

- Proof: see pp.34-48!

- What about \( \sigma < 1 \)?
  - zero long-run capital tax is correct
  - **but** convergence may take many hundred years
  - to be expected for \( \sigma \approx 1 \) due to continuity
Optimal Time Paths for $k_t$ and $\tau_t$

Left panel: $k_t$, Right panel: $\tau_t$

Figure 1: Optimal time paths over 300 years for capital stock (left panel) and wealth taxes (right panel) for various value of $\sigma$. Note: tax rates apply to gross returns not net returns, i.e. they represent an annual wealth tax.
$\sigma < 1$: Years until $\tau_t < 1\%$
Intuition

• In long-run, why is optimal $\{\tau_t\}$ increasing when $\sigma > 1$ and decreasing when $\sigma < 1$?

• Guess what? **Income and substitution effects**!

• Warm-up exercise: consider unexpected higher future taxation 
  $(1 + r_{t+1})(1 - \tau_{t+1})$ 
  - $\sigma > 1$: income effect dominates $\Rightarrow C_t \downarrow, a_{t+1} \uparrow$  
  - $\sigma < 1$: substitution effect dominates $\Rightarrow C_t \uparrow, a_{t+1} \downarrow$  
  - $\sigma = 1$: income & subst. effects cancel $\Rightarrow C_t, a_{t+1}$ constant

• One objective of optimal tax policy: high $k_t \Rightarrow$ high output, high tax base

  $\Rightarrow$ want to encourage savings $a_{t+1}$
  - $\sigma > 1$: income effect dominates $\Rightarrow$ want $\tau_{t+1} \geq \tau_t$
  - $\sigma < 1$: substitution effect dominates $\Rightarrow$ want $\tau_{t+1} \leq \tau_t$
  - $\sigma = 1$: income & subst. effects cancel $\Rightarrow$ want $\tau_t$ constant
Effect of Redistributive Preferences $\gamma$

Left panel: $k_t$, Right panel: $\tau_t$

Figure 3: Optimal time paths over 300 years for capital stock (left panel) and wealth taxes (right panel) for various redistribution preferences (zero represents no desire for redistribution; see footnote 16).
Linearized Dynamics

- Straub and Werning also analyze linearized system
  - see their Proposition 4
  - linearize around zero-tax steady state (i.e. Judd’s st. st.)
  - same tools as in Lecture 4 but 4-dimensional system (2 states, 2 co-states)
  - careful: they use “saddle-path stable” to refer to system of 2 states, i.e. “no. of negative eigenvalues = 1” or system is unstable except for knife-edge initial conditions \((k_0, C_{-1})\)

- Analysis confirms numerical results
Capital Taxation without Redistribution

- So far: capital taxation in environment with **redistributive** motif (capitalists and workers as in Judd, 1985)

- Different question: if government has to finance a flow of expenditure $g$, how should it raise the revenue?
  - capital taxes?
  - labor taxes?

- This is the question asked in Chamley (1986)
  - $\Rightarrow$ Ramsey taxation in **representative agent** model

- Won’t cover this case in detail
  - logic of Ramsey problem same: max. utility s.t. allocation $= CE$ with taxes
  - see Chamley (1986), Atkeson et al. (1999) among others, and Straub and Werning (2014, Section 3)
  - here: brief intuitive discussion
Capital Taxation without Redistribution

• Key to results in rep. agent models is thinking about “supply of capital” and its elasticity (responsiveness to rate of return)

• inelastic in short-run, elastic in long-run

• In standard growth model, consider $k_t(r_t, ...)$
  - supply at $t = 0$:
    $$k_0 = \hat{k}_0 \Rightarrow \text{elasticity} = 0$$
  - supply as $t \to \infty$:
    $$r^* = \frac{1}{\beta} - 1 \Rightarrow \text{elasticity} = \infty$$
    (if decrease $r$ by $\varepsilon$, $k_t \to 0$; if increase $r$ by $\varepsilon$, $k_t \to \infty$)

• “Infinite elasticity in long-run” prediction a bit extreme
  - relies on time-separability of preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$
  - but “more elastic in long-run than in short-run” is very general
Capital Taxation without Redistribution

• What does “more elastic in long-run than in short-run” imply for capital taxation?
  • motif for “front-loading” capital taxes: tax more today, than tomorrow
  • Chamley: no upper bounds on capital taxes ⇒ capital tax ⇒ 0 as $t \to \infty$
  • in fact, time-separable preferences + no bounds on taxes ⇒ all taxation at $t = 0$

• Werning and Straub point to extreme assumption: no upper bound on capital taxation
  • bounds ⇒ less front-loading
  • bounds may even bind indefinitely, i.e. capital taxes $> 0$ in long-run
Takeaway on Capital Taxation

- **Robust prediction:** if possible, want to tax more today than tomorrow
- **Not robust:** this implies that capital taxes should be zero in long-run