Lecture 9: Adding Growth to the Growth Model

ECO 503: Macroeconomic Theory I

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Adding Growth to the Growth Model

- Version of growth model we studied so far predicts that growth dies out relatively quickly.

- In reality, economies like U.S. have growth at $\approx 2\%$ per year for more than a century.
Adding Growth to the Growth Model

• What is missing?

• Consensus: technological progress

• Today: consequences of adding technological progress to growth model

• Deeper and important issue: how model the process that leads to technological progress
  • probably in later lecture (“endogenous growth models”)
  • what we do today will be very “reduced form”
Adding Growth: Choices

• Previously

\[ y_t = F(k_t, h_t) \]

• Let \( A_t \) = index of technology
  • increase in \( A_t \) = technological progress

• 3 different ways to “append” \( A_t \) into our existing model

\[
\begin{align*}
y_t &= A_t F(k_t, h_t) \quad \text{neutral} \\
y_t &= F(A_t k_t, h_t) \quad \text{capital augmenting} \\
y_t &= F(k_t, A_t h_t) \quad \text{labor augmenting}
\end{align*}
\]

• Note: if \( F \) is Cobb-Douglas, all three are isomorphic

• **Result:** to generate balanced growth, require that technological progress be labor augmenting

• Note: assumption is that tech. progress can be modeled as one-dimensional
  • simplifying assumption, tech. change takes many forms
  • recent work goes beyond this
Growth Model with Tech. Progress

- **Preferences:**
  \[ \sum_{t=0}^{\infty} \beta^t u(c_t) \]

- **Technology:**
  \[ y_t = F(k_t, A_t h_t), \quad \{A_t\}_{t=0}^{\infty} \text{ given} \]
  \[ c_t + i_t = y_t \]
  \[ k_{t+1} = i_t + (1 - \delta)k_t \]

- **Endowment:** \( k_0 = \hat{k}_0 \), one unit of time each period

- **Assumption:** path of technological change is known
  - can extend to stochastic growth model
  - will likely do this in second half of semester

- Now redo everything we did before
Social Planner’s Problem

\[
\max_{\{c_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t) \quad \text{s.t.}
\]

\[
k_{t+1} = F(k_t, A_t) + (1 - \delta)k_t - c_t
\]

\[
c_t \geq 0, \quad k_t \geq 0, \quad k_0 = \hat{k}_0
\]

- proceed as before \(\Rightarrow\) necessary and sufficient conditions

\[
u'(c_t) = \beta u'(c_{t+1})(F_k(k_t, A_t) + 1 - \delta)
\]

\[
k_{t+1} = F(k_t, A_t) + (1 - \delta)k_t - c_t
\]

\[+ \text{TVC} + k_0 = \hat{k}_0.\]
Asymptotic Behavior

- Looking for steady state as before does not really make sense
- Consider special case: $A_t$ grows at constant rate

$$A_{t+1} = (1 + g)A_t, \quad A_0 \text{ given, } 0 < g < \bar{g}$$

where $\bar{g}$ is an upper bound (more on this later)

- Idea is not that $A_t$ literally grows at constant rate ... 
- ... rather that trend growth is constant
  - what would things look like if trend growth were the only component?
Balanced Growth Path

• **Definition:** a balanced growth path (BGP) solution to the SP problem is a solution in which all quantities grow at constant rates

• In principle different variables could grow at different rates

• But rates turn out to be the same. To see this, consider

\[ c_t = F(k_t, A_t) + (1 - \delta)k_t - k_{t+1} \]

• For RHS to grow at constant rate, \( k_t \) has to grow at same rate as \( A_t \) \( \Rightarrow \) \( c_t \) also grows at same rate
Balanced Growth Path

• Now return to full necessary conditions for growth model

\[
\frac{u'(c_t)}{\beta u'(c_{t+1})} = F_k(k_t, A_t) + 1 - \delta \tag{*}
\]

\[
c_t = F(k_t, A_t) + (1 - \delta)k_t - k_{t+1} + TVC + \text{initial condition}
\]

• Looking for solution of form

\[
k_t^* = (1 + g)^t k_0^* \tag{**}
\]

i.e. need to find \(k_0^*\) such that this condition holds for all \(t\)

• **Important:** similar to steady state, a BGP is a \(k_0\) such that “if you start there, you stay there” (up to trend \(1 + g\))
  
  • “balanced growth” a.k.a. “steady state growth”
  
  • put differently: steady state in previous version of growth model = BGP with \(g = 0\)
Balanced Growth Path

• Now return to full necessary conditions for growth model

\[
\frac{u'(c_t)}{u'(c_{t+1})} = \frac{\beta u'(c_t)}{\beta u'(c_{t+1})} = F_k(k_t, A_t) + 1 - \delta \tag{\ast}
\]

\[
c_t = F(k_t, A_t) + (1 - \delta)k_t - k_{t+1}
\]

+ TVC + initial condition

• If (\ast\ast) holds, then RHS of (\ast) is constant (because CRS \( \Rightarrow F_k(k_t, A_t) = F_k(k_t/A_t, 1) \))

\[\Rightarrow \text{LHS of (\ast)} \text{ must also be constant}\]

• But \( c_{t+1}^* = (1 + g)c_t^* \). So how can we guarantee that \( \frac{u'(c_t)}{\beta u'(c_{t+1})} \) is constant with \( c_{t+1}^* = (1 + g)c_t^* \)? See next slide.
Balanced Growth Path

- Suppose
  \[ u(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \quad \text{(CRRA)} \]

- Then \( u'(c_t) = c_t^{-\sigma} \) and
  \[ \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{1}{\beta} \left( \frac{c_t^*}{c_{t+1}^*} \right)^{-\sigma} = \frac{1}{\beta} (1 + g)^\sigma \]

- \( \Rightarrow \) if \( u \) satisfies (CRRA), LHS of (*) is constant

- Still need to find \( k_0^* \)
  - We said LHS is constant, RHS is constant
  - still need to make them equal \( \Rightarrow \)
  \[ \frac{1}{\beta} (1 + g)^\sigma = F_k(k_0^*, A_0) + 1 - \delta \]
Balanced Growth Path

- Previous slide: \textbf{if } \( u \) satisfies (CRRA), \textbf{then } there is a BGP solution

- Turns out that (CRRA) is the only choice of utility function that works

- i.e. there is a BGP solution \textbf{if and only if } \( u \) satisfies (CRRA)
Balanced Growth Path

• **Only if** part: note that we require

\[
\frac{u'(c)}{u'(c(1 + g))} = \text{constant for all } c
\]

• Differentiate w.r.t. \( c \)

\[
u''(c) = (1 + g)u''(c(1 + g))\text{constant}
\]

\[
= (1 + g)u''(c(1 + g))\frac{u'(c)}{u'(c(1 + g))}
\]

\[
\frac{u''(c)c}{u'(c)} = \frac{u''(c(1 + g))c(1 + g)}{u'(c(1 + g))}
\]

\[
\frac{u''(c)c}{u'(c)} = a \quad (= \text{constant})
\]

\[
\frac{d \log u'(c)}{d \log c} = a \quad \Rightarrow \quad \log u'(c) = b + a \log c
\]

Hence \( u'(c) = e^b c^a = \text{monotone transformation of (CRRA)} \)
Balanced Growth Path

- From now on restrict preferences to (CRRA)
- Need ("−1 term" in (CRRA) doesn’t matter)

\[ \sum_{t=0}^{\infty} \beta^t \frac{(c_t^*)^{1-\sigma}}{1-\sigma} = \frac{(c_0^*)^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta(1 + g)^{1-\sigma})^t < \infty \]

- Need \( \beta(1 + g)^{1-\sigma} < 1 \)
- If \( \sigma < 1 \), need upper bound \( g < \bar{g} = \beta^{1/\sigma-1} - 1 \)
Balanced Growth Path

- Note: along a BGP, have $c_t, k_t, y_t$ all growing at same rate

- But

\[
\frac{i_t}{y_t} = \frac{k_t}{y_t} = \text{constant}
\]

- Same property as steady state in version without growth (see Lecture 7)

- = justification for thinking of U.S. economy in post-war period on a BGP
Transforming Model with Growth into Model without Growth

- Know how to solve for BGP = generalization of steady state
- But what about transition dynamics? Turns out this is easy:
  - transform model with growth into model without growth
  - analysis of transformed model same as before

- Preferences:
  \[ \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \]

- Technology:
  \[ c_t + k_{t+1} = F(k_t, A_t) + (1 - \delta)k_t \]

- Define detrended consumption and capital
  \[ \tilde{c}_t = \frac{c_t}{(1 + g)^t}, \quad \tilde{k}_t = \frac{k_t}{(1 + g)^t} \]
Transforming Model with Growth into Model without Growth

- **Preferences:**

\[
\sum_{t=0}^{\infty} \left( \beta (1 + g)^{1-\sigma} \right)^t \frac{\tilde{c}_t^{1-\sigma} - 1}{1 - \sigma} + \text{additive term}
\]

- **Technology:**

\[
\tilde{c}_t (1 + g)^t + \tilde{k}_{t+1} (1 + g)^{t+1} = F(\tilde{k}_t (1 + g)^t, A_0(1 + g)^t) + (1 - \delta)\tilde{k}_t (1 + g)^t
\]

\[
\tilde{c}_t + \tilde{k}_{t+1} (1 + g) = f(\tilde{k}_t) + (1 - \delta)\tilde{k}_t
\]

where we normalized \( A_0 = 1 \) and used that CRS \( \Rightarrow \)

\[
F(\tilde{k}_t (1 + g)^t, (1 + g)^t) = (1 + g)^t F(\tilde{k}_t, 1) = (1 + g)^t f(\tilde{k}_t)
\]
Transforming Model with Growth into Model without Growth

• Hence it is sufficient to solve (drop ∼’s for simplicity)

\[
\max_{\{c_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta_t \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \quad \text{s.t.}
\]

\[
c_t + k_{t+1}(1 + g) = f(k_t) + (1 - \delta)k_t
\]

where \(\tilde{\beta} = \beta(1 + g)^{1-\sigma}\)

• need \(\beta(1 + g)^{1-\sigma} < 1\)

• same restriction as before
Transforming Model with Growth into Model without Growth

• Everything else just like before. E.g. Euler equation

\[ c_t^{-\sigma} = \tilde{\beta} c_{t+1}^{-\sigma} \frac{f'(k_{t+1}) + 1 - \delta}{1 + g} \]

• Steady state

\[ \frac{1}{\tilde{\beta}} = \frac{f'(k^*) + 1 - \delta}{1 + g} \quad \Leftrightarrow \quad \frac{1}{\beta} (1 + g)^\sigma = f'(k^*) + 1 - \delta \]

• Steady state in transformed economy = BGP in original economy
  - transformed economy: plot log \( \tilde{k}_t \) against \( t \)
  - original economy: plot log \( k_t \) against \( t \): BGP = linear slope

\[ \log k_{t+1} - \log k_t = \log \left( \frac{k_{t+1}}{k_t} \right) = \log(1 + g) \approx g \]
FIGURE 1.1 Per Capita GDP in Seven Countries, 1870–2000
Prevailing Paradigm
for thinking about growth across countries

• Most countries **share a long run growth rate**
  - for these countries, policy differences have **level effects**
  - countries “transition around” in world BGP

• In terms of growth model
  - countries $i = 1, ..., n$, each runs a growth model
  - productivities satisfy (note: no $i$ subscript on $g$)

\[
A_{it} = A_{i0}(1 + g)^t e^{\varepsilon_{it}}
\]

  - interpret $A_{it}$ more broadly than technology, also include institutions, policy
  - every now and then, country gets $\varepsilon_{it}$ shock, triggers transition

• **Is prevailing paradigm = right paradigm?**
  - hard to say given data span only $\approx 100$ years
  - also recall from Lecture 7: transitions too fast rel. to data
Competitive Equilibria and BGP Prices

- Both ADCE and SOMCE can be defined just like before
- Prices along BGP
  \[ w^*_t = A_t F_h(k^*_t, A_t) \] grows at rate \( g \)
  \[ r^*_t = F_k(k^*_t, A_t) - \delta \] constant
- Easy to show: interest rate \( r^*_t \) satisfies
  \[ 1 + r^*_t = \frac{1}{\beta} (1 + g)^\sigma \]
- Will often see this written in terms of \( \rho = 1 / \beta - 1 \)
  \[ 1 + r^*_t = (1 + \rho)(1 + g)^\sigma \]
  \[ r^*_t \approx \rho + \sigma g \]
  where \( \approx \) uses \( \log(1 + x) \approx x \) for \( x \) small
- In continuous time, \( r^*_t = \rho + \sigma g \) exactly