Problem Set 2
Due Wednesday February 19

1. Suppose there are two consumers and two goods:
   \[ u_i(x_{1i}, x_{2i}) = x_{11} + u_2(x_{12}, x_{22}) = x_{22}; \ i \in \{1, 2\} \]
   \[ X_1 = X_2 = \mathbb{R}_+^2; \omega_i \geq 0, \ i = 1, 2. \]
   a. Prove that the Walrasian equilibrium allocation is independent of the distribution
      of initial endowments.
   b. For which values of \( \omega_i, i = 1, 2 \) does a Walrasian equilibrium exist? For those
      cases where the equilibrium exists, characterize equilibrium prices.

2. A cinnamon factory is located on an island with \( N \) households. Everybody on the
   island enjoys the taste of cinnamon but detests the cinnamon smell emitted by the
   factory. Let \( y_2 \) be the output of cinnamon and let \( s \) be the amount of smell emitted
   by the factory; \( y_1 \) denotes the input. The technology of the firm is given by
   \[ Y = \{(y_1, y_2, s): y_1 + y_2 = 0; y_1 + s = 0\} \]. The government has assigned rights to fresh
   air to every household. Hence, if the firm wishes to emit \( s \) units of smell it has to
   purchase the corresponding amount of air rights from every household. Each
   household is endowed with 1 unit of air rights and \( \omega_{1i} = 10 \) units of good 1. By \( x_{3i} \)
   we denote consumer i’s consumption of air-rights. The set of feasible consumptions
   is \( X = \{(x_1, x_2, x_3): x_1 \geq 0, x_2 \geq 0, 1 \leq x_3 \geq 0\} \). The tastes of households are
   described by the utility function \( u_i(x_{1i}, x_{2i}, x_{3i}) = x_{3i} \ln x_{2i} + (x_{3i} - 1) / N \)
   a. What is the commodity space for this economy? Define the technology of the firm
      incorporating the regulation of the government.
   b. Normalize the price of good 1 to be 1. Define a competitive equilibrium and solve
      for the equilibrium quantities and prices.
   c. It is often argued that as the number of agents increases, the price taking
      assumption inherent in the definition of Walrasian equilibrium becomes more
      reasonable. Suppose we would increase the number of households in the village
      and suppose that the factory was in fact an industry consisting of many small
      firms. Does price taking become a more reasonable assumption as the number of
      agents in the economy goes to infinity? Explain.

3. Consider the following economy with \( I \) consumers and uncertainty. There are \( S \) states
   of nature, \( s \in \{1, \ldots, S\} \) and one physical good. The endowment of consumer \( i \) in state
   \( s \) is \( \omega_{si} \geq 0 \). Assume that preferences for consumer \( i \) are represented by the von
   Neumann-Morgenstern utility function \( U_i(x_{1i}, \ldots, x_{Si}) = \sum_{s=1}^{S} \pi_s u_i(x_{si}) \) where
   \( u_i: \mathbb{R}_+ \to \mathbb{R} \) is strictly increasing and strictly concave and differentiable. Assume
   that there is no "aggregate risk" in the economy in the sense that \( \sum_{i=1}^{I} \omega_{si} \) is
   independent of \( s \).
   a. Commodities are distinguished by the state of nature they can be consumed in.
      Define the commodity space and a competitive equilibrium for this economy.
b. Show that in a Pareto optimal allocation every consumer is perfectly insured in the sense that \( x_{si} \) is independent of \( s \) for all \( i \).

c. Characterize equilibrium prices.

4. Consider the following two-good exchange economy. There are two types of consumers and \( n \) consumers of each type. Both types have the utility function \( u(x, y) = \min \{2x, y\} + \min \{x, 2y\} \). Consumers of type 1 have an endowment of \( \omega_1 = (10,0) \) and consumer type 2 have endowment \( \omega_2 = (0,10) \).

a. Find the competitive equilibrium allocations and prices.

b. Suppose \( n = 1 \). Let \( 0 \leq z < 10 \). Find a value for \( z \) such that the consumer of type 1 is better off in the competitive equilibrium of an economy where he has the smaller endowment \( (z,0) \) and consumer 2's endowment stays unchanged than in the original economy. How do you interpret this finding?

c. Suppose \( n = 10 \). Show that for each \( z \in [0,10) \) a consumer of type 1 is better off in the competitive equilibrium of the original economy than in an economy where he has the smaller endowment \( (z,0) \) and all other consumer's endowments stay unchanged. How do you interpret this finding?

d. Suppose instead, the utility function is: \( u(x, y) = \min \{x, y\} \). How would part (c) change?

5. Suppose an exchange economy consists of \( I \) households, with \( u_i(m_i, x_i) = m_i + v_i(x_i), x_i \in \mathbb{R}^+_i, m_i \in \mathbb{R} \); where \( m_i \) denotes the money paid/received by household \( i \). The functions \( v_i \) are concave. Households can buy and sell arbitrarily large quantities of money. Each household has a zero endowment of money and a non-negative endowment of non-money goods \( \omega_i \). Aggregate endowment is denoted by \( \overline{\omega} = \sum_{i=1}^{I} \omega_i \).

Define

\[
g(\overline{\omega}) = \max_{x \geq 0} \sum_{i=1}^{I} v(x_i) \]

subject to \( \sum_{i=1}^{I} x_i = \omega \)

a. Show that any Pareto optimal allocation of the non-money commodities \( x = (x_1, \ldots, x_I) \) must satisfy \( \sum_{i=1}^{I} v_i(x_i) = g(\overline{\omega}) \).

b. Show that \( g \) is concave.

c. Assume that \( g \) is differentiable. Show that \( (1, p) \) is a Walrasian equilibrium price of the economy if \( p \in \partial g(\overline{\omega}) \) where \( \partial g(\overline{\omega}) \) denotes the vector of partial derivatives (the gradient vector) of \( g \) at \( \overline{\omega} \).