Problem Set 5

1. Consider the following 3 period model $t = 0, 1, 2$. In period 1 the state is either low or high and similarly in period 2 the state is either low or high. There are two assets: a riskfree bond and equity. The riskfree bond pays off 1 in period 2 independent of the state of nature. Equity has the following payoffs in period 2:

- 2 if the state is high in periods 1 and 2;
- 1.5 if the state is high in period 1 and low in period 2;
- 1 if the state is low in period 1 and high in period 2;
- 0 if the state is low in both periods.

Both assets have no payoffs in periods zero or one. However, in periods 0, 1 and 2 both assets are traded. The riskfree asset has a price of 1 in every period. Equity has a price of 1 in period 0; a price of 0.5 in period 1 if the state is low and a price of 1.7 in period 1 if the state is high.

a. Find the period 0 price of a call option on equity that can be exercised only in period 2 and has a strike price of 0.8.

a. Suppose the option may be exercised in any period. What is the price of the option in that case?

2. Let $r_m$ be the ‘pricing asset’ defined in class. We have established that

$$q_k = E(\tilde{r}_m \cdot \tilde{r}_k) = E(\tilde{r}_m) \cdot E(\tilde{r}_k) + \text{cov} (\tilde{r}_m, \tilde{r}_k)$$

Show that this equation implies the asset pricing formula:

$$\bar{\rho}_k - \bar{\rho}_s = \beta_{km} (\bar{\rho}_m - \bar{\rho}_s)$$

where $\rho_k = \tilde{r}_k / q_k$ denotes a random variable describing the (normalized) returns of asset $k$; $\rho_s = \tilde{r}_s / q_s$ denotes the returns of the riskfree asset ($r_i = (1, ..., 1)$); $\bar{\rho}_s = E(\rho_s)$ is the expected return and $\beta_{km} := \text{cov} (\rho_k, \rho_m) / \text{var} (\rho_m)$.

3. a. Give an example of a risk averse utility function that is not variance averse.

b. Suppose $u : [0, 1]^s \to \mathbb{R}$ is a strictly increasing, continuous and variance averse utility function. Let $\pi_s$ denote the probability of state $s$. Show that the utility function is risk averse, that is: if we replace a “lottery” (state dependent consumption vector) $x = (x_1, ..., x_s)$ with $x = (\sum_s x_s \pi_s, ..., \sum_s x_s \pi_s)$ (which is the expected value in each state) then the agent is better off.

4. Consider the following economy with two goods. There is one buyer and two sellers and two states of nature, $s = 1, 2$. In state $s$ the buyer has the utility function

$$u_s(x_1, x_2) = \beta_1 x_1 + x_2, \quad x_1 \geq 0, x_2 \geq 0 \quad \beta_1 = 2, \beta_2 = 1;$$

the two sellers have the (state-independent) utility function

$$v_s(x_1, x_2) = \alpha x_1 + x_2, \quad x_1 \geq 0, x_2 \geq 0, \alpha \geq 0.$$

Independent of the state, each seller is endowed with 1 unit of good 1 and 10 units of good 2 and
the buyer is endowed with 0 units of good 1 and 10 units of good 2. The probability of the states is given by $\pi_1 = \pi_2 = 1/2$. The sellers receive a signal that reveals the state of nature. The buyer receives no information.

(a) Give the range of $\alpha$ for which there is a fully revealing rational expectations equilibrium. For the given range find a fully revealing rational expectations equilibrium.

(b) Find the range of $\alpha$ for which there is both a fully revealing rational expectations equilibrium and a rational expectations equilibrium that is not fully revealing. For the given range, provide an example of a rational expectations equilibrium that is not fully revealing.

5. A production path $y = (y_1, ..., y_t, ...)$ maximizes joint profit if

$$\sum_{t=0}^{\infty} p_t y_{b,t} + p_{t+1} y_{a,t} \geq \sum_{t=0}^{\infty} p_t \tilde{y}_{b,t} + p_{t+1} \tilde{y}_{a,t},$$

for any other production path $\tilde{y} = (\tilde{y}_1, ..., \tilde{y}_t, ...)$ . Suppose that the production path $y$ is bounded, and prices satisfy $\sum_t p_t < \infty$.

a. Show that if $y$ maximizes joint profit, then $y$ is efficient.

b. Show that if $y$ is short-run profit maximizing, then it maximizes joint profit.

6. Consider the following one-sector growth model: $V(c) = \sum_{t=0}^{\infty} (1/2^t) \ln(c_t)$ and $Y = \{-k, 0, c, k' | x + k' \leq \alpha k\}$. The consumer's endowment is $\omega = (1, 0, 0, ...)$.

a. For which value of $\alpha$ is there an equilibrium with bounded consumption and production path. Find the equilibrium for the given range.

b. Consider values of $\alpha$ for which there is no bounded equilibrium. If we drop the boundedness requirement from the definition of equilibrium, is there a competitive equilibrium? If so, find it.

c. Can you change the commodity space in such a way that a bounded equilibrium exists for all values of $\alpha$?