Problem 1

Consider an economy with two commodities. Give an example of a function $z(p)$ defined on $P_{\varepsilon} = \{(p_1, p_2) | \varepsilon < \frac{p_1}{p_2} < \frac{1}{\varepsilon}\}$ and with values in $\mathbb{R}^2$ that is continuous, homogeneous of degree zero and satisfies Walras’ law, and cannot be generated from a rational preference relation.

Take $z(p) = \left( \frac{p_1}{p_2} - 1, -\frac{p_1}{p_2} \left( \frac{p_2}{p_1} - 1 \right) \right)$. Obviously this is continuous, homogeneous of degree zero, and satisfies Walras’ law. However, it cannot be generated by a rational preference relation. To see this, consider the following two price vectors:

$$(p_1, p_2) = \left( 1, \frac{3}{2} \right); \quad (p'_1, p'_2) = \left( 1, \frac{1}{2} \right).$$

Then $z(p) = \left( -\frac{1}{3}, \frac{2}{9} \right)$, and $z(p') = (1, -2)$, so

$$p \cdot z(p') = \left( 1, \frac{3}{2} \right) \cdot (1, -2) = -2,$$

$$p' \cdot z(p) = \left( 1, \frac{1}{2} \right) \cdot \left( -\frac{1}{3}, \frac{2}{9} \right) = -\frac{2}{9}.$$

So $z(\cdot)$ violates the weak axiom, and hence cannot be generated from rational preferences.

Problem 2

Consider a consumer with strictly increasing, concave utility function and a strictly positive endowment in each good. Let $z(p)$ denote the excess demand. Provide an example, in which for some price sequence $p^n \gg 0$, $p^n \to p$ with $p \neq 0$ and $p_t = 0$ and $z_2(p^n)$ stays bounded.

Let $L = 3$, $\omega = (1, 1, 1)$ and $u(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Let $p^n = (1, p^n_2, p^n_3)$ be such that $1 > p^n_2 > p^n_3 > 0$ and $p^n_2 \to 0$. Then $p^n \to p = (1, 0, 0)$ and $x(p^n) = (0, 0, \frac{p^n_2 \omega}{p^n_3})$, so $z_2(p^n) = -1$ is bounded as desired.
Problem 3

Consider the following economy. There are two periods, two states in the second period, and two consumers. There is one physical commodity. The endowment in period 0 is 2 for both consumers. Consumer 1 has an endowment of 1 in both states in period 1 and consumer 2 has an endowment of 2 in both states in period 1. Let $x_0, x_{1s}$ denote (respectively) consumption in period 0, period 1 state $s$.

$$u_1(x_0, x_{11}, x_{12}) = u_2(x_0, x_{11}, x_{12}) = x_0 + x_{1s} - \frac{x_{1s}^2}{4}$$

There is one asset that pays off $(\alpha, 1 - \alpha)$ units of the good, where $\alpha \in (0, 1)$.

(a) What are the Pareto optimal allocations in this economy?

First define the set of feasible allocations:

$$A = \{x \in \mathbb{R}_+^2 : x_0^4 + x_0^2 \leq 4 \text{ and } x_{1s} + x_{1s}^2 \leq 2 \text{ for } s = 1, 2\}.$$  

Clearly this set is convex. In addition, define the set of utilities possibilities:

$$U = \{(u_1, u_2) : \exists x \in A : u_i \leq u(x_0, x_{11}, x_{12}) \text{ for } i = 1, 2\}.$$  

$U$ is closed (by continuity of the utility functions) and bounded above. Since the utility functions are concave, and the consumption sets convex, then $U$ is also convex (see MWG exercise 16.E.2). The conditions of Proposition 16.E.2 in MWG (page 560) are met, and hence we can fully characterize the Pareto frontier of this economy using welfare weights. That is, for each Pareto optimal allocation $x^*$ there exists a vector of welfare weights $(\lambda_1, \lambda_2)$ such that $x^*$ solves

$$\max_{(x_1, x_2) \in A} \sum_{i=1, 2} \lambda_i \left(x_i^0 + x_{1s} - \frac{(x_{1s})^2}{4} + x_{1s}^2 - \frac{(x_{1s})^2}{4}\right).$$

The Pareto optimal allocation must maximize some weighted sum of utilities, subject to the constraints $x_0^4 + x_0^2 = 4, x_{11}^4 + x_{11}^2 = 3, x_{12}^4 + x_{12}^2 = 3$, and non-negativity constraints on all variables. The F.O.C.’s are:

$$x_0^1 : [1 - \lambda = 0, x_0 = (0, 4)] \text{ or } [1 - \lambda < 0, x_0 = 0] \text{ or } [1 - \lambda > 0, x_0 = 4] \quad (1)$$

$$x_{11}^1 : 1 - \frac{1}{2} x_{11}^1 + \lambda \left(-1 + \frac{3}{2} - \frac{1}{2} x_{11}^1\right) = 0 \Leftrightarrow x_{11}^1 = \frac{2 + \lambda}{1 + \lambda} \quad (2)$$

$$x_{12}^1 : x_{12}^1 = \frac{2 + \lambda}{1 + \lambda} \quad (3)$$

(For condition (2), note that the corner solutions - $[(2 + \lambda) - (1 + \lambda)x_{11}^1 < 0, x_{11}^1 = 0]$ or $[(2 + \lambda) - (1 + \lambda)x_{11}^1 > 0, x_{11}^1 = 3]$ are impossible, so we get an interior solution; the F.O.C.’s for $x_{12}^1$ are identical).

This yields three types of Pareto optimal allocations:
(b) What is the asset market equilibrium? Can you Pareto rank equilibria as a function of $\alpha$?

Let $q$ denote the asset price in period 0 (normalizing the price of the good to 1). Then individual $i$ (with period 1 endowment $(\omega_i, \omega_i)$) solves

$$\max_{z_i} (2 - qz_i) + (\omega_i + \alpha z_i) - \frac{(\omega_i + \alpha z_i)^2}{4} + (\omega_i + (1 - \alpha)z_i) - \frac{(\omega_i + (1 - \alpha)z_i)^2}{4}.$$ 

The F.O.C. is

$$-q + \frac{1}{2} \alpha (\omega_i + \alpha z_i) + (1 - \alpha) - \frac{1}{2} (1 - \alpha) (\omega_i + (1 - \alpha)z_i) = 0,$$

$$-q + 1 - \frac{1}{2} \omega_i - \frac{1}{2} (\alpha^2 + (1 - \alpha)^2) z_i = 0,$$

$$z_i = \frac{2 - \omega_i - 2q}{(\alpha^2 + (1 - \alpha)^2)}.$$ 

So $z_1(q) = \frac{1 - 2q}{\alpha^2 + (1 - \alpha)^2}$, $z_2(q) = \frac{-2q}{\alpha^2 + (1 - \alpha)^2}$. For asset markets to clear, we need $z_1(q) + z_2(q) = 0$, implying an equilibrium asset price of $q = \frac{1}{4}$; this yields equilibrium asset purchases of $z_1^* = \frac{1}{2(\alpha^2 + (1 - \alpha)^2)}$, $z_2^* = \frac{-1}{2(\alpha^2 + (1 - \alpha)^2)}$. Then consumption levels are

Consumer 1: $x_0^1 = 2 - \frac{1}{8(\alpha^2 + (1 - \alpha)^2)}$, $x_{11}^1 = 1 + \frac{\alpha}{2(\alpha^2 + (1 - \alpha)^2)}$, $x_{12}^1 = 1 + \frac{(1 - \alpha)}{2(\alpha^2 + (1 - \alpha)^2)}$

Consumer 2: $x_0^2 = 2 + \frac{1}{8(\alpha^2 + (1 - \alpha)^2)}$, $x_{11}^2 = 2 - \frac{\alpha}{2(\alpha^2 + (1 - \alpha)^2)}$, $x_{12}^2 = 2 - \frac{(1 - \alpha)}{2(\alpha^2 + (1 - \alpha)^2)}$

implying utility levels

Consumer 1: $4 + \frac{-1 + 4\alpha + 4(1 - \alpha)}{8(\alpha^2 + (1 - \alpha)^2)} - \frac{1}{4} \left(1 + \frac{\alpha}{2(\alpha^2 + (1 - \alpha)^2)}\right)^2 - \frac{1}{4} \left(1 + \frac{1 - \alpha}{2(\alpha^2 + (1 - \alpha)^2)}\right)^2$

$= \frac{7}{2} + \frac{1}{16(\alpha^2 + (1 - \alpha)^2)}$

Consumer 2: $4 + \frac{1}{16(\alpha^2 + (1 - \alpha)^2)}$

Note that both $u^1$ and $u^2$ are maximized at $\alpha = \frac{1}{4}$, and that both consumers become worse off as $\alpha$ moves further away from $\frac{1}{4}$. The shape of the functions is:
So the answer is that we can Pareto-rank equilibria with $\alpha$. Let $E(\alpha)$ denote the equilibrium allocation as a function of $\alpha$; let $a \succ b$ denote the binary relation "$a$ Pareto-dominates $b$" (and every consumer is strictly better off in equilibrium $a$ than in $b$), and we have:

$$E(\alpha) \succ E(\alpha') \iff |\alpha - \frac{1}{2}| < |\alpha' - \frac{1}{2}|$$

**Note:** The above solution assumed that with only one good, consumers must consume whatever they have in period 1 (i.e., that $x^{i}_{11} = \omega^i + \alpha z^i$, $x^{i}_{12} = \omega^i + (1 - \alpha)z^i$). If inequality constraints are allowed, the solution remains unchanged. To see this, note first that period 1 preferences have a satiation point at 2 (in both states); so, $x^{i}_{11} < \omega^i + \alpha z^i$ is optimal iff $\omega^i + \alpha z^i > 2$ - in which case $x^{i}_{11} = 2$ is the optimal consumption level. Similarly, a slack budget constraint in state 2 is optimal iff $\omega^i + (1 - \alpha)z^i > 2$, in which case $x^{i}_{12} = 2$ is optimal. Next, note that there is no equilibrium in which $\omega^i + \alpha z^i > 2$ and $\omega^i + (1 - \alpha)z^i > 2$ (for either consumer $i$); buying this much of the asset can only be optimal if $q = 0$ (otherwise, decreasing $z^i$ a bit would leave period 1 consumption unchanged at 2, while increasing period 0 consumption); but then both consumers will want to consume 2 units of the good in both states in period 1, violating feasibility (aggregate endowment is only 3). Since consumer 2’s endowment is $\omega^2 = 2$, this also implies that there is no equilibrium with $z^2 > 0$; hence, consumer 2’s budget constraint will always be satisfied with equality: then as above, his demands are $z^2 = \frac{-2q}{\alpha^2 + (1 - \alpha)^2}$, $x^{11}_{11} = 2 - \frac{2q}{\alpha^2 + (1 - \alpha)^2}$, $x^{12}_{12} = 2 - \frac{2q(1 - \alpha)}{\alpha^2 + (1 - \alpha)^2}$.

Suppose first that there is an equilibrium where consumer 1’s budget constraint is slack in state 1 - so $1 + \alpha z^1 > 2 > 1 + (1 - \alpha)z^1$ (requires $\alpha > \frac{1}{2}$); then his optimal consumption levels in period 1 will be $x^{11}_{11} = 2$, $x^{12}_{12} = 1 + (1 - \alpha)z^1$, so the optimal demand for $z^1$ is

$$z^1 = \arg \max_{z^1} \left[ 2 - qz^1 + 1 + (1 - \alpha)z^1 - \frac{(1 + (1 - \alpha)z^1)^2}{4} \right] = \frac{1}{1 - \alpha} - \frac{2q}{(1 - \alpha)^2}$$

For this to satisfy $1 + \alpha z^2 > 2$, we need

$$\frac{\alpha}{1 - \alpha} - \frac{2q}{(1 - \alpha)^2} > 1 \Leftrightarrow 2q < \alpha(1 - \alpha) - (1 - \alpha)^2 \quad (*)$$

4
Feasibility for good 1 requires

\[ x_{11}^1 + x_{11}^2 = 2 + 2 - \frac{2\alpha q}{\alpha^2 + (1 - \alpha)^2} \leq 3 \iff 2\alpha q \geq \alpha^2 + (1 - \alpha)^2 \quad (**) \]

Since \( \alpha(1 - \alpha) - (1 - \alpha)^2 \) reaches a maximum of \( \frac{1}{8} \), while \( \alpha^2 + (1 - \alpha)^2 \geq \frac{1}{2} \) for \( \alpha \in [0, 1] \), it’s impossible to satisfy (*) and (**) together; hence, no such equilibrium. Similarly, feasibility for good 2 is violated in any “equilibrium” with \( 1 + (1 - \alpha)z^i > 2 > 1 + \alpha z^i \). So, the solution must be as above, where \( x_{11}^i = \omega^i + \alpha z^i, x_{12}^i = \omega^i + (1 - \alpha)z^i \).
Problem 4

Consider an economy with one physical good, S states. Assume that \( \sum_i \omega_{si} = \omega \forall s \). Further, assume that there are \( I \leq S \) households and that the endowments \((\omega_1, ..., \omega_I)\) are linearly independent vectors. All consumers have von Neumann Morgenstern utility functions and are risk averse.

(a) Suppose the asset structure is complete. What is the equilibrium allocation?

As shown in question 3 of Problem set 2, the equilibrium allocation is state-independent: consumer \( i \) consumes \( x_i^s = \sum_s \pi_s \omega_{si} \) in every state. We can define

\[
\alpha_i \equiv \sum_s \pi_s \frac{\omega_{si}}{\omega} \Rightarrow x_i^s = \alpha_i \omega \forall s.
\]

Notice, moreover, that \( \sum_i \alpha_i = 1 \).

(b) How many assets are necessary so that the allocation in (a) is an equilibrium allocation? Find an asset structure that is minimal in the sense that it contains as few assets as possible and has the allocation in (a) as an equilibrium.

Let \( K \) be the minimal number of assets, with return matrix \( R \), and let \( x \) be an equilibrium allocation. For \( x \) to be an asset equilibrium allocation, there must exist a vector \((z_1, ..., z_I) \in \mathbb{R}^{KI}\) of portfolios which finance \( x \):

\[
\begin{pmatrix}
    x_{i1} \\
    \vdots \\
    x_{iS}
\end{pmatrix}
= \begin{pmatrix}
    \omega_{i1} \\
    \vdots \\
    \omega_{Si}
\end{pmatrix} + \begin{pmatrix}
    r_{11} & \cdots & r_{1K} \\
    \vdots & \ddots & \vdots \\
    r_{S1} & \cdots & r_{SK}
\end{pmatrix} \begin{pmatrix}
    z_{1i} \\
    \vdots \\
    z_{Ki}
\end{pmatrix} \quad \forall i.
\]

From part (a),

\[
\begin{pmatrix}
    x_{i1}^* \\
    \vdots \\
    x_{Si}^*
\end{pmatrix}
= \begin{pmatrix}
    \omega_{i1} \\
    \vdots \\
    \omega_{Si}
\end{pmatrix} \quad \forall i.
\]

Combining,

\[
\begin{pmatrix}
    \omega_{i1} \\
    \vdots \\
    \omega_{Si}
\end{pmatrix} + \begin{pmatrix}
    r_{11} & \cdots & r_{1K} \\
    \vdots & \ddots & \vdots \\
    r_{S1} & \cdots & r_{SK}
\end{pmatrix} \begin{pmatrix}
    z_{1i} \\
    \vdots \\
    z_{Ki}
\end{pmatrix} = \begin{pmatrix}
    \alpha_i \omega_{i1} \\
    \vdots \\
    \alpha_i \omega_{Si}
\end{pmatrix} \quad \forall i.
\]

We can write this for the economy as

\[
\begin{pmatrix}
    \omega_{11} & \cdots & \omega_{1I} \\
    \vdots & \ddots & \vdots \\
    \omega_{S1} & \cdots & \omega_{SI}
\end{pmatrix} + RZ = \begin{pmatrix}
    \alpha_1 \omega_{11} & \cdots & \alpha_I \omega_{11} \\
    \vdots & \ddots & \vdots \\
    \alpha_1 \omega_{S1} & \cdots & \alpha_I \omega_{SI}
\end{pmatrix} \Rightarrow RZ = \begin{pmatrix}
    \alpha_1 \omega_{11} - \omega_{11} & \cdots & \alpha_I \omega_{11} - \omega_{1I} \\
    \vdots & \ddots & \vdots \\
    \alpha_1 \omega_{S1} - \omega_{S1} & \cdots & \alpha_I \omega_{SI} - \omega_{SI}
\end{pmatrix}.
\]
It will be possible to ensure that the equilibrium allocation is feasible for arbitrary endowment vectors \((\omega_i)_{i \in I}\) if the rank of the matrix \(RZ\) is at least as large as the rank of the matrix \(J\).

Suppose there are \(K\) assets.

Then, since \(\text{Rank}(A \cdot B) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}\) and \(\text{Rank}(Z) \leq K\), a necessary condition is that \(K\) be at least as large as the rank of \(J\).

The claim is that the rank of \(J\) is exactly \(I - 1\).

\textbf{Claim 1} \(\text{Rank}(J) < I\)

\textbf{Proof.} Add across the columns of \(J\):

\[\forall s : \sum_i \alpha_i \omega - \sum_i \omega_i s = \omega \sum_i \alpha_i - \omega = 0.\]

The facts that \(\sum_i \omega_i s = \omega\) and \(\sum_i \alpha_i = 1\) were explained above. The implication is that the columns are linearly dependent, hence the rank of the matrix must be strictly smaller than \(I\). \(\blacksquare\)

\textbf{Claim 2} \(\text{Rank}(J) = I - 1\)

\textbf{Proof.} Suppose by way of contradiction that \(\text{Rank}(J) < I - 1\). Then there must be \(I - 1\) columns of \(J\) which are linearly dependent. For ease of exposition consider removing any column \(i\) from \(J\), so that:

\[J' = \begin{pmatrix}
\alpha_1 \omega - \omega_1 & \cdots & \alpha_{i-1} \omega - \omega_{1,i-1} & \alpha_{i+1} \omega - \omega_{1,i+1} & \cdots & \alpha_I \omega - \omega_{1}\n\vdots & \ddots & \vdots & \vdots & & \vdots \\
\alpha_1 \omega - \omega_{S1} & \cdots & \alpha_{i-1} \omega - \omega_{S,i-1} & \alpha_{i+1} \omega - \omega_{S,i+1} & \cdots & \alpha_I \omega - \omega_{S}\n\end{pmatrix}.
\]

Then the contradiction hypothesis is that there exists some vector of weights \((\lambda_j)_{j \neq i}\) with at least some \(\lambda_j \neq 0\) such that the weighted sum of the columns of \(J'\), with weights \((\lambda_j)_{j \neq i}\), returns a null vector:

\[\sum_{j \neq i} \lambda_j \begin{pmatrix}
\alpha_j \omega - \omega_{1j} \\
\vdots \\
\alpha_j \omega - \omega_{Sj}
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix},\]

\[\Rightarrow \sum_{j \neq i} \lambda_j \begin{pmatrix}
\omega_{1j} \\
\vdots \\
\omega_{Sj}
\end{pmatrix} = \sum_{j \neq i} \lambda_j \begin{pmatrix}
\alpha_j \omega \\
\vdots \\
\alpha_j \omega
\end{pmatrix} = \begin{pmatrix}
k \\
\vdots \\
k
\end{pmatrix}. \tag{1}
\]

Notice that the terms with \(\alpha_j \omega - \text{ in the middle of expression (1) - are independent of s.}

7
This is a contradiction to linear independence of the endowment vectors. To see why, take any $s$ and use $\sum_i \omega_{si} = \omega$ together with $\sum_{j \neq i} \lambda_j \omega_{sj} = \kappa$:

$$
\sum_{j \neq i} \lambda_j \omega_{sj} = \kappa \equiv \frac{\kappa}{\omega} \cdot \sum_i \omega_{si} \equiv \frac{\kappa}{\omega} \cdot \left( \omega_{si} + \sum_{j \neq i} \omega_{sj} \right),
$$

$$
\Rightarrow \omega_{si} = \sum_{j \neq i} \left( \lambda_j \frac{\omega}{\kappa} - 1 \right) \omega_{sj},
$$

$$
\Rightarrow \omega_i = \sum_{j \neq i} \left( \lambda_j \frac{\omega}{\kappa} - 1 \right) \omega_j.
$$

And therefore $\omega_i$ is a linear combination of $\left( \omega_{j} \right)_{j \neq i}$, which violates linear independence. Consequently, no $I-1$ columns of $I$ can be linearly dependent, and therefore $\text{Rank} \left( I \right) = I - 1$.  

For a minimal asset structure, consider a set of $I - 1$ assets, where asset $r_i$ pays $x_i^s - \omega_{si}$ in state $s$ (where $x_i^s = \sum s \pi s \omega_{si}$), and each asset has price zero. If consumer $i \in \left\{ 1, ..., I - 1 \right\}$ buys one unit of asset $i$, then his consumption in each state $s$ is $x_i^s$; if consumer $I$ sells one unit of each asset, then his consumption in each state $s$ is

$$
x_{sI} = \omega_{sI} - \sum_{i=1}^{I-1} (x_i^s - \omega_{si}) = \overline{\omega} - \sum_{i=1}^{I-1} x_i^s = \overline{\omega} - (\overline{\omega} - x_I^s) = x_I^s.
$$

So, each consumer $i$ can replicate his consumption in (a) with asset $r_i$ (and consumer $I$ with asset $-\sum_{i=1}^{I-1} r_i$). Furthermore, doing so is optimal, since he also could have afforded the consumption generated by any other asset $r_j$ in the original economy, but did not find it optimal (i.e. in the original economy, equilibrium prices were such that $p \cdot (x_i^s - \omega_i) = 0$; so consumer $i$ could have added any consumption vector $(x_j^s - \omega_j)$ to his own bundle $x_i^s$ at zero cost).

A more abstract way of proving the claim (you may skip to the next problem, but I just wanted to keep what prior year TAs have done).

Go back to

$$
\left(\begin{array}{c} \omega_{1i} \\ \vdots \\ \omega_{Si} \end{array}\right) + \left(\begin{array}{ccc} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{Si1} & \cdots & r_{SK} \end{array}\right) \left(\begin{array}{c} z_{1i} \\ \vdots \\ z_{Ki} \end{array}\right) = \left(\begin{array}{c} x_i^s \\ \vdots \\ x_i^s \end{array}\right) = \left(\begin{array}{c} \sum s \pi s \omega_{si} \\ \vdots \\ \sum s \pi s \omega_{si} \end{array}\right) \forall i.
$$

Re-write this for the aggregate economy as

$$
\Omega + RZ = \Pi \Omega, \text{ or } RZ = (\Pi - I_S) \Omega
$$

(2)
where

\[
R = \begin{pmatrix}
  r_{11} & \cdots & r_{1K} \\
  \vdots & \ddots & \vdots \\
  r_{s1} & \cdots & r_{sK}
\end{pmatrix},
Z = \begin{pmatrix}
  z_{11} & \cdots & z_{1I} \\
  \vdots & \ddots & \vdots \\
  z_{K1} & \cdots & z_{KI}
\end{pmatrix},
\]

\[
\Pi = \begin{pmatrix}
  \pi_1 & \cdots & \pi_S \\
  \vdots & \ddots & \vdots \\
  \pi_1 & \cdots & \pi_S
\end{pmatrix},
\Omega = \begin{pmatrix}
  \omega_{11} & \cdots & \omega_{1I} \\
  \vdots & \ddots & \vdots \\
  \omega_{s1} & \cdots & \omega_{sI}
\end{pmatrix}.
\]

In the second expression in (2), the rank of the LHS is less than or equal to rank(R) \leq K. We will next show that the rank of the RHS is at least I - 1, which will imply that \( K \geq I - 1 \), as desired. Note that \( \Pi \) is idempotent, so

\[
(\Pi - I_S)^2 = \Pi^2 - 2I_SI_S + I_S = I_S - \Pi.
\]

Then \( \text{rank}(\Pi - I_S) = \text{tr}(I_S - \Pi) = S - \sum \pi_S = S - 1 \). Therefore, the null space of \( \Pi - I_S \), \( \text{Null}(\Pi - I_S) = \{ y \in \mathbb{R}^S : (\Pi - I_S)y = 0 \} \) is a 1-dimensional subspace of \( \mathbb{R}^S \), i.e. there is \( v \in \mathbb{R}^S \setminus \{0\} \) such that \( \text{Null}(\Pi - I_S) = \text{span}\{v\} = \{\alpha v : \alpha \in \mathbb{R}\} \). If \( v \in \text{span}\{\omega_1, \ldots, \omega_I\} \), then \( v = \sum_{j=1}^I \alpha_j \omega_j \) for some unique set of coefficients \( \alpha_1, \ldots, \alpha_I \), where uniqueness follows from the linear independence of \( \omega_1, \ldots, \omega_I \). Since \( v \neq 0 \), there is \( j \in \{1, \ldots, I\} \) such that \( \alpha_j \neq 0 \). Then, \( v \notin \text{span}\{\omega_1, \ldots, \omega_I\} \setminus \{\omega_j\} \) by uniqueness of the coefficients. If \( v \notin \text{span}\{\omega_1, \ldots, \omega_I\} \), then we can arbitrarily choose \( j \in \{1, \ldots, I\} \) and we still have \( v \notin \text{span}\{\omega_1, \ldots, \omega_I\} \setminus \{\omega_j\} \). Note that \( (\Pi - I_S)\Omega = [(\Pi - I_S)\omega_1, \ldots, (\Pi - I_S)\omega_I] \). We will now show that \( \{(\Pi - I_S)\omega_i : i \neq j\} \) is linearly independent proving that \( \text{rank}(\Pi - I_S) \geq |\{(\Pi - I_S)\omega_i : i \neq j\}| = I - 1 \). Suppose not, then there exist \( \alpha_i \in \mathbb{R} \) for \( i \neq j \) such that \( \sum_{i \neq j} \alpha_i = 1 \) and \( 0 = \sum_{i \neq j} \alpha_i(\Pi - I_S)\omega_i = (\Pi - I_S)\sum_{i \neq j} \alpha_i \omega_i \). So \( \sum_{i \neq j} \alpha_i \omega_i \in \text{Null}(\Pi - I_S) \), moreover \( \sum_{i \neq j} \alpha_i \omega_i \neq 0 \) by linear independence of \( \omega_i \)'s. Therefore there is a scalar \( \beta \neq 0 \) such that \( \sum_{i \neq j} \alpha_i \omega_i = \beta v \). Then \( v = \sum_{i \neq j} \alpha_i \omega_i \in \text{span}\{\omega_1, \ldots, \omega_I\} \setminus \{\omega_j\} \), a contradiction.

\footnote{Recall: if \( A^2 = -A \), then \( \text{rank}(A) = \text{tr}(-A) \), where the trace of a matrix is the sum of the diagonal elements.}
Problem 5

Consider the assets $r_1, ..., r_K$. Assume that $r_k \neq 0$. Let $q = (q_1, ..., q_K)$ be no-arbitrage prices. Further assume that there is a portfolio $z^*$ such that $Rz^* \geq 0$, $Rz^* \neq 0$.

(a) Show that there is a vector of state prices $\mu = (\mu_1, ..., \mu_S)$ such that $q = \mu \cdot R$. Are the state prices positive? Give a proof or a counter-example.

Compared to MWG, we are dropping the requirement that $r_k \geq 0$. The natural question is what could go wrong with that. It is clear that the part of the proof of Lemma 19.E.1 is in page 702 of the book remains unchanged. We can still define

$$V = \{ v \in \mathbb{R}^S : v = Rz \text{ for some } z \in \mathbb{R}^K \text{ with } q \cdot z = 0 \},$$

and invoke the separating hyperplane theorem to assert that there must exist a vector $\mu$ (interpreted as the "state prices") such that $\mu$ separates $V$ from $\mathbb{R}_+^S$.

The problem arises when we want to assert that asset prices must be proportional to $\mu' \cdot R$. This argument is in page 703: The fact that $R \geq 0$ implies (since it is always possible to choose $\mu \geq 0$) that $\mu' \cdot R \geq 0$ (and $\neq 0$), and this is used to rule out the possibility of $q$ not being proportional to $\mu' \cdot R$. In the absence of $r_k \geq 0$ (let alone $\mu \geq 0$), it is no longer necessarily the case that $\mu' \cdot R \neq 0$, and the previous argument no longer follows.

A different method which is not from convexity theory will be used here to sidestep this issue and prove the existence of state prices.

By no arbitrage, $q \cdot z^* > 0$. Consider any two portfolios $x$, $x'$ such that $Rx = Rx'$. Then it must be the case that $qx = qx'$. To see this, suppose by way of contradiction that $qx < qx'$. Then selling short one unit of portfolio $x'$ and purchasing one unit of portfolio $x$ generates a positive surplus of funds. Use those surplus to purchase an amount $\alpha_2$ of portfolio $z^*$ such that $q \cdot (\alpha_2 z^* + x - x') = 0$. $\alpha_2$ will be strictly positive, so this composite portfolio, which costs 0, contains a strictly positive amount of $z^*$. The paysoffs from the portfolio equal

$$R \cdot (\alpha_2 z^* + x - x') = \alpha_2 Rz^* + Rx' - Rx = \alpha_2 Rz^*.$$

$\alpha_2 Rz^* \geq 0$, and $\alpha_2 Rz^* \neq 0$. This is an arbitrage opportunity, since non-negative, non-zero return was achieved at a 0 cost.

Intuition: The existence of $z^*$ ensures that a price vector can only be arbitrage-free if whenever two portfolios generate the same payoffs ($R \cdot x$), they have the same value ($q \cdot x$). Notice that in the case studied in the book, with $r_k \geq 0$, the existence of such a portfolio will be easily satisfied (a portfolio entirely composed by any single asset will do). However, once $r_k$ may have negative entries, it does not necessarily follow that a portfolio such as $z^*$ (with non-negative, non-zero returns) will exist; and, consequently, $qx \neq qx'$ for $x$ and $x'$ with $Rx = Rx'$ may not be enough to generate an arbitrage opportunity – it will generate surplus funds, but the asset structure may be such that there is no way to use
those funds to generate non-negative, non-zero returns. For example the asset structure in exercise 6 below: There are $S$ states, only one asset, and this asset has positive payoffs in some states but negative payoffs in other states.

It is possible then to define a function mapping from state-contingent payoff vectors to portfolio values. The thrust of the previous paragraph is precisely that each state-contingent payoffs vector can have a unique monetary value; therefore this function is well defined.

Formally, let $V$ be the subspace of $\mathbb{R}^S$ spanned by the return matrix $R$:

$$V = \{ v \in \mathbb{R}^S : v = Rx \text{ for some } x \in \mathbb{R}^K \}.$$ 

$V$ is the set of state-contingent payoffs that can be generated with the asset structure $R$. We know from linear algebra that $V$ is a vector space:

1) $v \in V, \alpha \in \mathbb{R} \Rightarrow \alpha v \in V$ - because $\alpha v = R\alpha x$ and $\alpha x \in \mathbb{R}^K$.
2) $v, v' \in V \Rightarrow v + v' \in V$ - because $v + v' = R(x + x')$ and $x, x' \in \mathbb{R}^K$.

Then define the mapping $g : V \to \mathbb{R}$ as follows:

$$g(v) = q \cdot x \text{ for any } x \text{ such that } v = Rx.$$ 

$g$ is the function that assigns monetary values to state contingent payoffs $v \in \mathbb{R}^S$. Again, it is very possible that two different vectors $x$ and $x'$ are such that $v = Rx = Rx'$. However, we have established that $g$ will be uniquely defined, since $qx = qx'$. It is only state-contingent payoffs (and not the portfolios with which those payoffs were actually achieved) which matter to evaluate portfolios.

Next, we establish two properties of $g$:

(Notice: I will prove the properties here, but in case you have doubts about what's going on there is an explanation below.)

$$\alpha \in \mathbb{R}, y \equiv \alpha x \Rightarrow g(\alpha Rx) \equiv g(Ry) = qy \equiv \alpha qx = \alpha g(Rx),$$

$$x, x' \in \mathbb{R}^K, y \equiv x + x' \Rightarrow g(Rx + Rx') = qy = q(x + x') = qx + qx' = g(Rx) + g(Rx')$$

(3) and (4) are the defining properties of linear functions. That is, $g$ will satisfy (3) and (4) if and only if it is a linear function; and since it satisfies them, it is so. And we know what linear functions look like in $\mathbb{R}^S$: they are the inner product. So, there is an $S$-dimensional vector $\mu$ such that

$$g(Rx) = \mu \cdot Rx.$$ 

And therefore, since $g(Rx) = q \cdot x$, we get

$$\forall x \in \mathbb{R}^K : q \cdot x = \mu \cdot Rx,$$

$$\Rightarrow q = \mu R.$$
State prices need not be positive (though it is always possible to find non-negative state prices). Consider the following counter-example, with 3 assets, 3 states, and the following state prices and asset returns:

\[
q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}
\]

Note that \(q\) is no-arbitrage:

\[
Rz = \begin{pmatrix} \frac{z_1}{z_2 + 2z_3} \\ \frac{z_2}{z_2 + 2z_3} \\ -\frac{2z_3}{z_2 + 2z_3} \end{pmatrix},
\]

so \(Rz \geq 0\) requires \(z_2 + 2z_3 = 0\); then \(Rz \geq 0, Rz \neq 0 \Rightarrow z_1 > 0 \Rightarrow q \cdot z = z_1 > 0\). State prices solve:

\[
q = (\mu' \cdot R)' = R' \cdot \mu \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 - \mu_3 \\ 2(\mu_2 - \mu_3) \end{pmatrix}.
\]

So \((\mu_1, \mu_2, \mu_3) = (1, \alpha, \alpha)\), where \(\alpha\) is any real number (not necessarily positive).

Appendix: Detailed explanation of properties (3) and (4). There is no additional information here, so if you have understood the proof above you don’t need to read this.

Recall that

\[
\bar{V} = \{v \in \mathbb{R}^S : v = Rx \text{ for some } x \in \mathbb{R}^K\},
\]

\[
g(v) = q \cdot x \text{ for any } x \text{ such that } v = Rx. \tag{5}
\]

\[
g(\alpha v) = \alpha g(v) \tag{3}
\]

\[
g(v + v') = g(v) + g(v') \tag{4}
\]

1) Take any \(v \in \bar{V}\), any scalar \(\alpha\), and take any \(x\) such that \(v = Rx\) (remember that this is the condition for \(v \in \bar{V}\)). Notice that if we define \(y \equiv \alpha x\), then we can write \(\alpha v = \alpha Rx = R\bar{y}\). Recall that \(\alpha v \in \bar{V}\). So, because \(y\) is such that \(\alpha v = R\bar{y}\), the definition of the function \(g\) in (5) implies \(g(\alpha v) = q \cdot y\). Therefore, property (3) has been proven:

\[
g(\alpha v) = q \cdot y = q \cdot \alpha x = \alpha q \cdot x = \alpha g(v).
\]

2) The proof of (4) is very similar. Take any two vectors \(v, v' \in \bar{V}\), and two vectors \(x, x' \in \mathbb{R}^K\) such that \(v = Rx\) and \(v' = Rx'\) (again, these must exist since \(v, v' \in \bar{V}\), and also \(v + v' \in \bar{V}\)). Define \(y \equiv x + x'\), and then we can write \(v + v' = Rx + Rx' = R\bar{y}\). So, because \(y\) is such that \(v + v' = R\bar{y}\), the definition of the function \(g\) in (5) implies \(g(v + v') = q \cdot y\). Therefore, property (4) has been proven:

\[
g(v + v') = q \cdot y = q \cdot (x + x') = q \cdot x + q \cdot x' = g(v) + g(v').
\]
The two properties imply that \( g \) is linear: there is an \( S \)-dimensional vector \( \mu \) such that
\[
g(v) = \mu \cdot v. \tag{6}
\]

So take any \( x' \in \mathbb{R}^K \) and let \( v' = Rx' \). By the definition of \( g \) in (5), \( g(v') = q \cdot x' \). Together with (6), we conclude that
\[
q \cdot x' = g(v') = \mu \cdot v' = \mu \cdot Rx'.
\]

Since this must hold for all \( x' \in \mathbb{R}^K \), it must be the case that \( q = \mu \cdot R \).

(b) Suppose there is no portfolio \( z \) such that \( Rz \geq 0, Rz \neq 0 \) and \( q \) is no-arbitrage. Give an example for which there are no state prices \( \mu = (\mu_1, ..., \mu_S) \) such that \( q = \mu \cdot R \).

Let
\[
R = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 2 \\
0 & -1 & -2
\end{bmatrix}.
\]

Then the return to a portfolio \((z_1, z_2, z_3)\) is
\[
\begin{bmatrix}
z_1 \\
-z_1 + (z_2 + 2z_3) \\
-(z_2 + 2z_3)
\end{bmatrix}.
\]

Note first that there is no \( z \) such that \( Rz \geq 0, Rz \neq 0 \). (For the return in state 3 to be non-negative, we need \( z_2 + 2z_3 \leq 0 \). If this inequality is strict, then a non-negative return in state 2 requires \( z_1 < 0 \), in which case the return in state 1 is strictly negative. If \( z_2 + 2z_3 = 0 \), then the return is either zero in all states (if \( z_1 = 0 \), or strictly negative in one of states 1, 2). This implies that any asset price vector \( q \) is arbitrage-free. So, pick \( q = (0, 1, 1) \) : then state prices must satisfy
\[
\begin{bmatrix}
0 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
\mu_1 & \mu_2 & \mu_3
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 2 \\
0 & -1 & -2
\end{bmatrix} = \begin{bmatrix}
\mu_1 - \mu_2 & \mu_2 - \mu_3 & 2(\mu_2 - \mu_3)
\end{bmatrix}.
\]

So we need \( \mu_2 - \mu_3 = 1 \) and \( 2(\mu_2 - \mu_3) = 1 \), obviously impossible.

(Note: Many finance textbooks use an alternative definition of no-arbitrage: namely, “\( q \) is no-arbitrage if (i) the MWG condition holds; AND (ii) there is no \( z \) s.t. \( q \cdot z < 0 \) and \( Rz = 0 \). Under this definition, no such example exists: a theorem in Duffie, “Dynamic Asset Pricing Theory”, proves that a state price vector exists if and only if \( q \) is no-arbitrage, with no restrictions on return matrix \( R \).)

(c) In part (a), show that the state price vector is unique if and only if the asset structure is complete.
If the asset structure is complete, then the matrix \( R \) has full rank, so the equation \( q^T = \mu \cdot R \) has a unique solution \( \mu \). If the asset structure is not complete, then the counterexample in (a) (where state prices were anything of the form \((1, \alpha, \alpha)\), and the rank of the return matrix was only 2) establishes that the state price vector may not be unique.

(d) Suppose the asset structure is complete. Are the state prices positive? Give a proof or a counter-example.

They are positive. Proof: if the asset structure is complete, so return matrix \( R \) has full rank, then

\[ Rz_s = e_s \]

has a solution for all \( s \) (i.e. \(- \forall s, \) there exists a portfolio \( z_s \) with payoff 1 in state \( s \), 0 everywhere else). By no-arbitrage, equilibrium prices must satisfy \( q \cdot z_s > 0 \forall s \). Then the state price vector \( \mu \) satisfies

\[ \mu_s = \mu \cdot e_s = \mu \cdot Rz_s = q \cdot z_s > 0 \forall s \]

So \( \mu >> 0 \).
Problem 6

Consider the following economy with two states, and one asset that has returns

\[ r = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

There is one physical good and both agents own one unit of the good in each state. The utility functions are

\[ U_1(x_1, x_2) = \frac{1}{4} x_1 + \frac{3}{4} x_2, \]
\[ U_2(x_1, x_2) = \frac{3}{4} \ln x_1 + \frac{1}{4} \ln x_2. \]

Does this economy have an asset market equilibrium? If yes, find one. If no, show that there does not exist one.

Consider agent 1. Since the marginal utility of consumption in state 2 is larger than that of consumption in state 1, he would like to transfer as much wealth as possible to state 2. This is achieved by buying as much of the asset as possible. Consequently, the price \( q \) of the asset must be strictly positive; otherwise unboundedly long positions would be budgetarily feasible for consumer 1.

When prices are strictly positive, feasibility for agent 1 (i.e., that \( qz^1 \leq 0 \) can only obtain by choosing \( z^1 \leq 0 \). This requires that consumer 2’s desired (and feasible) purchases of the asset be \( z^2 \geq 0 \).

Hence an equilibrium, if it exists, must have \( q > 0 \) and \( z^2 \geq 0 \).

Next look at consumer 2. His problem is

\[ V(q) = \max_{z^2} \frac{3}{4} \ln x_1 + \frac{1}{4} \ln x_2 \]
\[ \text{s.t. } qz^2 \leq 0, x_1 = 1 - z^2, x_2 = 1 + z^2 \]

Replace for \( x_1 \) and \( x_2 \) in the objective function to re-write:

\[ V(q) = \max_{z^2} \frac{3}{4} \ln (1 - z^2) + \frac{1}{4} \ln (1 + z^2) \]
\[ \text{s.t. } qz^2 \leq 0 \]

It is straightforward to show that the global, unconstrained (given the admissible transfers of wealth across states) optimum occurs at \( z^2 = -1/2 \). Therefore if prices are strictly positive (as they must be at an equilibrium), the consumer is able to achieve his unconstrained optimum by shortselling exactly 1/2 units of the asset.

But this is incompatible with the requirement that \( z^2 \geq 0 \).

In conclusion, no equilibrium exists in this incomplete markets economy where agents ”disagree” about the probabilities of states (if we interpret utility functions as representing expected utility preferences). (This is not a general statement – equilibria will exist in other examples of economies with disagreement among the agents about probabilities.)