Optimal Execution:
IV. Heterogeneous Beliefs and Market Making

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The Agents

Market Maker

- Nasdaq definition: agent that places competitive orders on both sides of the order book in exchange for privileges.
- Acts as a scaled-down version of the market.
- In this lecture: **Liquidity provider**, someone who posts an order book/transaction cost curve.
- Strategy: adapt pricing by *reading client flows*.

Clients

- In this lecture: **Liquidity takers**, agents who trade with the Market maker.
- Are *information* driven.
Theoretical literature

- **Early approaches**: Hasbrouck(2007), Chakrborti - Toke - Patriarca - Abergel(2011)
- **Inventory models**: Garman(1976), Amihud - Mendelson(1980)
Propose a **stochastic, agent-based** model in which existence and (**tractable** and **realistic**) properties of the LOB appear as a result of the analysis (**not as hypotheses**)

**Client model**

- Summarize sparsely the link between trade and price dynamics.

**Market maker model**

- **Tractable** market making strategy based on previous result.

R.C. - K. Webster (2012)
Setup: heterogeneous beliefs

Let

1. \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with \(W\) a \(\mathbb{P}\)-BM that generates \(\mathbb{F}\).

2. \(\mathbb{F}^k \subset \mathbb{F}\) generated by a \(P\)-BM \(W^k\).

3. \(P^k\) s.t. \(P^k |_{\mathcal{F}_t^k} \sim \mathbb{P} |_{\mathcal{F}_t^k}\).

4. \(P_t\) an Itô process adapted to all \((\mathbb{F}^k)_{k=0\ldots n}\).

5. In \(L^2\) and a.s. \(P_t\) grows polynomially in \(t\).

NB

Each agent has his /her own distinct filtration and probability measure. They are potentially mutually exclusive, but the price process is adapted to all of them.
Anatomy of a trade

- **Midprice** $P_t$ announced by the market at time $t$
- **Market maker** proposes a **transaction cost curve** $c_t(l)$ around $P_t$
- Market maker cannot differentiate clients **pre-trade**
- Client triggers a **trade** of volume $l_t$
- Client obtains volume $l_t$ and pays **cash flow** $P_t l_t + c_t(l_t)$.
- Market maker tries to identify clients **post-trade**
Agents behaviors

- Market maker controls transaction cost function \( l \mapsto c_t(l) \).
- Client \( i \) controls trading volumes/speeds \( I_i \).

Hypotheses

1. Marginal costs are defined: \( c \in C^1 \).
2. Clients may choose not to trade, \( c_t(0) = 0 \) and the midprice is well defined, \( c'_t(0) = 0 \).
3. Marginal costs increase with volume: \( c_t \) is convex.
4. \( c_t \) has compact support.
Legendre transform

\[ \gamma_t(\alpha) := \sup_{l \in \text{supp}(c_t)} (\alpha l - c_t(l)) \]

Duality

c_t convex with compact support \iff \gamma''_t is a positive finite measure.

The distribution \( \gamma''_t \) represents the order book formed by the orders of the market maker.
Client model

Client’s Objective

▶ Summarize sparsely the link between trade and price dynamics in a general, theoretical framework.

*Not* trying to build a optimal trading strategy.

Assumptions

▶ The client only tries to *predict*, not *cause* price movements.
▶ The client’s decision does not affect $c_t$.

Realistic if the client is ’small enough’. 
Client model

- **Exogeneous state variables** $P_t$ and $c_t$ are Itô processes. $P_t$ has polynomial growth and $c_t$ convex with compact support.

- **Endogeneous state variables**

\[
\begin{align*}
\frac{dL_t^i}{dt} &= l_t^i dt \\
\frac{dX_t^i}{dt} &= L_t^i dP_t - c_t(l_t^i) dt
\end{align*}
\]

$L_t^i$ is the *total* position of the client. $X_t^i$ is his *wealth*, marked to the midprice. $l_t^i$, the rate at which he trades, is his *control*.

- **Objective function**

\[
J^i = E_{\mathbb{P}_i} \left[ U^i(X_{\tau^i}^i, p_{\tau^i}) \right]
\]

with $\tau^i$ a stopping time.
Optimal trading strategy

**Theorem**

Under suitable integrability assumptions on $U^i$ and $\tau^i$, the optimal strategy is

$$\alpha^i_t := c'_t(l^i_t) = \mathbb{E}_{Q^i} \left[ p_{\tau^i} - P_t \mid \mathcal{F}^i_t \right]$$

with

$$\frac{dQ^i}{dP^i} = \frac{\partial_x U^i(X^i_{\tau^i}, p_{\tau^i})}{\mathbb{E}_{P^i} \left[ \partial_x U^i(X^i_{\tau^i}, p_{\tau^i}) \right]}.$$
Testing the client model

Hypotheses

- Under $\mathbb{Q}^i$, $\tau^i \sim \exp(\beta^i)$ independent of $P_t$.
- $\sigma^i_t := |c'_t(l^i_t) - (p_{\tau^i} - P_t)| \leq \frac{\text{spread}}{2}$

This leads to a *two parameter* model linking trade to price dynamics: $(\beta^i, \sigma^i)$.

Testing the hypotheses on data

- Assume all clients have one of *two* time scales.
- choose $(\beta_1, \beta_2)$ that minimizes error between implied and realized alpha.
Source

- Nasdaq 'fullview' data: all public quotes, all trades, nanosecond timestamps.
- Long parsing time: Data goes from 7:00-10:00am.
Two time scales

- $L^1$ regression used.
- Time scales: 9 ($\approx 0.5$ seconds) and 158 ticks.
- Mean error: 0.026.
- Mean half-spread: 0.063.
- Lower bound on error: 0.005.
Market maker model

Market Maker’s Objectives

▶ Find a *tractable* market making strategy based on previous result.
▶ Build a theoretical model for the order book that *replicates* the empirical features described before.

Strategy

Exploit link between trade and price dynamics to dynamically adapt pricing.
Market maker model: endogenous variables

With primal variables

\[
\begin{align*}
\frac{dL_t}{dt} &= -\frac{1}{n} \sum_i l^i_t dt \\
\frac{dX_t}{dt} &= L_t dP_t + \frac{1}{n} \sum_i c_t(l^i_t) dt
\end{align*}
\]

With dual variables

\[
\begin{align*}
\frac{dL_t}{dt} &= -\frac{1}{n} \sum_i \gamma_t' (\alpha^i_t) dt \\
\frac{dX_t}{dt} &= L_t dP_t + \frac{1}{n} \sum_i [\alpha^i_t \gamma_t' (\alpha^i_t) - \gamma_t (\alpha^i_t)] dt
\end{align*}
\]

Assume the market maker is risk-neutral.
Model for the $\alpha^i_t$

- **Notation**
  We will denote by $\mu_t(\alpha)$ the client belief distribution, that is, the empirically observed distribution of the $(\alpha^i_t)$.

- **Microscopic model (SDE)**
  
  $$d\alpha^i_t = -\rho \alpha^i_t dt + \sigma dB^i_t + \nu dB_t$$

  mean reversion corresponds to decay of information.

- **Macroscopic model (SPDE)**
  
  $$d\mu_t(\alpha) = \left[ \frac{1}{2} (\sigma^2 + \nu^2) \Delta \mu_t(\alpha) + \rho \nabla (\alpha \mu_t(\alpha)) \right] dt - \nu \nabla \mu_t(\alpha) dB_t$$
Approximate model for $P_t$

- **Intuition**
  - Do not want to make an explicit model for the price process.
  - Instead, would like to *infer* the price from client trades.

- **Implied alpha relationship**

\[
\alpha_t^i := c_t'(l_t^i) = \mathbb{E}_{Q^i} \left[ \int_t^{\infty} e^{-\beta^i(t-s)} \, dp_s \right| \mathcal{F}_t^i]
\]

- **Estimator**

\[
dp_t^\lambda := \sum_{i=1}^{n} \lambda^i \left( \beta^i \alpha_t^i \, dt - d\alpha_t^i \right)
\]

with $\sum \lambda^i = 1$. 
Estimation result

**Entropic feedback**

There exists $\lambda$ s.t.

$$E \left| P_t - p_t^\lambda \right|^2 \leq \epsilon^2 \frac{1}{n} \sum_i E(Q^i, P) \approx -\epsilon^2 \int_0^t \left< \log \left( \frac{\gamma_s'}{\mu_s} \right), \mu_s \right> ds$$

with $E$ the *entropy* function and

$$\epsilon = \sqrt{\frac{n}{\sum (\sigma^i)^2}} \leq \frac{1}{n} \sum \sigma^i$$
Approximate control problem

State variables

\[
\begin{align*}
\frac{dL_t}{dt} &= -\langle \gamma'_t, \mu_t \rangle \, dt \\
\frac{d\mu_t(\alpha)}{dt} &= \left[ \frac{1}{2} \left( \sigma^2 + \nu^2 \right) \Delta \mu_t(\alpha) + \rho \nabla (\alpha \mu_t(\alpha)) \right] \, dt - \nu \nabla \mu_t(\alpha) dB_t
\end{align*}
\]

Objective function

\[
J^\lambda = \int_0^\infty e^{-\beta t} E \left[ L_t \langle id, (\beta \lambda)_t \rangle + \langle -L_t \beta id + (id - \bar{\alpha}_t) \gamma'_t - \gamma_t, \mu_t \rangle \right] \, dt
\]
under the constraint \( \int_0^\infty \left\langle e^{-\beta t} \log \left( \frac{\gamma''_t}{\mu_t} \right) , \mu_t \right\rangle \, dt \leq C. \)
Pontryagin

**BSDE**

The solution to the Pontryagin BSDE gives rise to the market maker’s ‘shadow alpha’:

$$\alpha^*_t = \langle \text{id}, \lambda_t + \frac{(\beta \lambda)_t - \beta \mu_t}{\beta + \rho} \rangle$$

**Hamiltonian**

$$\mathcal{H}(\gamma, \mu, \alpha^*) = \langle (\text{id} - \alpha^*)\gamma' - \gamma + \epsilon \log \gamma'' , \mu \rangle$$
Profitability of an order without feedback

Define

\[ m(\alpha) = (\alpha - \alpha^*) \cdot \left( \text{spread} \int_{\alpha}^{\infty} \mu \right) \text{ if } \alpha \geq 0 \]

then we have:

\[ \mathcal{H}(\gamma, \mu, \alpha^*) = \langle \gamma'', m \rangle + \epsilon \langle \log \gamma'', \mu \rangle \]

Optimal strategy with feedback

\[ \frac{\gamma''(\alpha)}{\mu(\alpha)} = \frac{\epsilon}{C - m(\alpha)} \]

where \( C \) is a renormalization constant.
Simulated example

**Figure:** Blue: Optimal order book $\gamma''$. Green: Client alpha distribution $\mu$. 