Recall the resource allocation problem:

\[
\begin{align*}
\text{maximize} & \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \quad \vdots \\
& \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& \quad x_1, x_2, \ldots, x_n \geq 0,
\end{align*}
\]

where

\[
\begin{align*}
c_j & = \text{profit per unit of product } j \text{ produced} \\
b_i & = \text{units of raw material } i \text{ on hand} \\
a_{ij} & = \text{units raw material } i \text{ required to produce one unit of product } j.
\end{align*}
\]
Closing Up Shop

If we produce one unit less of product \( j \), then, for each \( i \), we free up:

\[ a_{ij} \text{ units of raw material } i. \]

Selling these unused raw materials for \( y_1, y_2, \ldots, y_m \) dollars/unit yields
\[ a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \text{ dollars}. \]

Only interested if this revenue exceeds lost profit on each product \( j \):
\[ a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \ldots, n. \]

Consider a buyer offering to purchase our entire inventory.

Subject to above constraints, buyer wants to minimize cost:

\[
\begin{align*}
\text{minimize} & \quad b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{subject to} & \quad a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
& \quad a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
& \quad \vdots \\
& \quad a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
& \quad y_1, y_2, \ldots, y_m \geq 0.
\end{align*}
\]
Duality

Every Problem:

maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \]
\[ x_j \geq 0 \quad j = 1, 2, \ldots, n, \]

Has a Dual:

minimize \[ \sum_{i=1}^{m} b_i y_i \]
subject to \[ \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n \]
\[ y_i \geq 0 \quad i = 1, 2, \ldots, m. \]
Primal Problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m, \\
& \quad x_j \geq 0 \quad j = 1, \ldots, n.
\end{align*}
\]

Original problem is called the \textit{primal problem}.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual in “Standard” Form:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m} -b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j \quad j = 1, \ldots, n, \\
& \quad y_i \geq 0 \quad i = 1, \ldots, m.
\end{align*}
\]

Dual is “negative transpose” of primal.

Theorem Dual of dual is primal.
Weak Duality Theorem

**Theorem.** If \((x_1, x_2, \ldots, x_n)\) is feasible for the primal and \((y_1, y_2, \ldots, y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i.
\]

**Proof.**

\[
\begin{align*}
\sum_j c_j x_j & \leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j \\
& = \sum_{ij} y_i a_{ij} x_j \\
& = \sum_i \left( \sum_j a_{ij} x_j \right) y_i \\
& \leq \sum_i b_i y_i.
\end{align*}
\]
Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?

Answer is provided by the Strong Duality Theorem (coming later).
### Simplex Method and Duality

#### A Primal Problem:

**Objective Function**

<table>
<thead>
<tr>
<th>obj</th>
<th>0</th>
<th>+</th>
<th>-3</th>
<th>x1</th>
<th>+</th>
<th>2</th>
<th>x2</th>
<th>+</th>
<th>1</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>x1</td>
<td>-</td>
<td>-1</td>
<td>x2</td>
<td>-</td>
<td>2</td>
<td>x3</td>
</tr>
<tr>
<td>w2</td>
<td>3</td>
<td>-</td>
<td>-3</td>
<td>x1</td>
<td>-</td>
<td>4</td>
<td>x2</td>
<td>-</td>
<td>1</td>
<td>x3</td>
</tr>
</tbody>
</table>

**Notes:**
- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

#### Its Dual:

**Objective Function**

<table>
<thead>
<tr>
<th>obj</th>
<th>0</th>
<th>+</th>
<th>0</th>
<th>y1</th>
<th>+</th>
<th>-3</th>
<th>y2</th>
</tr>
</thead>
<tbody>
<tr>
<td>z1</td>
<td>3</td>
<td>-</td>
<td>0</td>
<td>y1</td>
<td>-</td>
<td>3</td>
<td>y2</td>
</tr>
<tr>
<td>z2</td>
<td>-2</td>
<td>-</td>
<td>1</td>
<td>y1</td>
<td>-</td>
<td>-4</td>
<td>y2</td>
</tr>
<tr>
<td>z3</td>
<td>-1</td>
<td>-</td>
<td>-2</td>
<td>y1</td>
<td>-</td>
<td>-1</td>
<td>y2</td>
</tr>
</tbody>
</table>

**Notes:**
- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: \(x_2\) enters, \(w_2\) leaves. Make analogous pivot in dual: \(z_2\) leaves, \(y_2\) enters.
Second Iteration

After First Pivot:

Primal (feasible):

\[
\begin{align*}
\text{obj} & = 3/2 + -3/2 \cdot x_1 + -1/2 \cdot w_2 + 1/2 \cdot x_3 \\
w_1 & = 3/4 - -3/4 \cdot x_1 - 1/4 \cdot w_2 - 9/4 \cdot x_3 \\
x_2 & = 3/4 - -3/4 \cdot x_1 - 1/4 \cdot w_2 - 1/4 \cdot x_3 
\end{align*}
\]

Dual (still not feasible):

\[
\begin{align*}
\text{obj} & = -3/2 + -3/4 \cdot y_1 + -3/4 \cdot z_2 \\
z_1 & = 3/2 - 3/4 \cdot y_1 - 3/4 \cdot z_2 \\
y_2 & = 1/2 - -1/4 \cdot y_1 - -1/4 \cdot z_2 \\
z_3 & = -1/2 - -9/4 \cdot y_1 - -1/4 \cdot z_2 
\end{align*}
\]

Note: **negative transpose property intact.**

Again, use primal to pick pivot: \( x_3 \) enters, \( w_1 \) leaves.

Make analogous pivot in dual: \( z_3 \) leaves, \( y_1 \) enters.
Primal:

- Is *optimal*.

Dual:

- Negative transpose property remains intact.
- Is *optimal*.

**Conclusion**

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.
Conclusion on previous slide is the *strong duality theorem* which we now state formally:

**Theorem.** *If the primal problem has an optimal solution,*

\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*), \]

*then the dual also has an optimal solution,*

\[ y^* = (y_1^*, y_2^*, \ldots, y_m^*), \]

*and*

\[ \sum_j c_j x_j^* = \sum_i b_i y_i^*. \]

**Paraphrase:**

If primal has an optimal solution, then there is *no duality gap.*
Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

Example of infinite gap:

\[
\begin{align*}
\text{maximize} & \quad 2x_1 - x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
Theorem. At optimality, we have

\[ x_j z_j = 0, \quad \text{for } j = 1, 2, \ldots, n, \]
\[ w_i y_i = 0, \quad \text{for } i = 1, 2, \ldots, m. \]
Proof

Rewrite the proof of the Weak Duality Theorem:

\[
\sum_j c_j x_j \leq \sum_j (c_j + z_j) x_j = \sum_j \left( \sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \leq \sum_i b_i y_i,
\]

The inequalities come from the fact that

\[
x_j z_j \geq 0, \quad \text{for all } j,
\]

\[
w_i y_i \geq 0, \quad \text{for all } i.
\]

By Strong Duality Theorem, the inequalities are equalities at optimality.
When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

*An Example.* Showing both primal and dual dictionaries:

Looking at dual dictionary: $y_2$ enters, $z_2$ leaves.

On the primal dictionary: $w_2$ leaves, $x_2$ enters.

After pivot...
Going in, we have:

| \( \text{obj} \) | \(-6.6667\) | \(+\) | \(-4.6667\) | \(+\) | \(-1.3333\) | \(+\) | \(0.0\) | \(+\) | \(-3.3333\) | \(+\) | \(x_4\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(w_1\) | \(-6.3333\) | \(-\) | \(0.3333\) | \(-\) | \(0.6667\) | \(-\) | \(0.0\) | \(-\) | \(-2.3333\) | \(-\) | \(x_4\) |
| \(x_2\) | \(1.6667\) | \(-\) | \(-0.6667\) | \(-\) | \(-0.3333\) | \(-\) | \(0.0\) | \(-\) | \(0.6667\) | \(-\) | \(x_4\) |
| \(w_3\) | \(3.0\) | \(-\) | \(4.0\) | \(-\) | \(1.0\) | \(-\) | \(3.0\) | \(-\) | \(0.0\) | \(-\) | \(x_4\) |

Looking at dual: \(y_1\) enters, \(z_4\) leaves.

Looking at primal: \(w_1\) leaves, \(x_4\) enters.
Dual Simplex Method Pivot Rule

Referring to the primal dictionary:

- Pick leaving variable from those rows that are infeasible.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...
Going in, we have:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>x4</th>
<th>x2</th>
<th>w3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-15.7143</td>
<td>2.7143</td>
<td>-0.1429</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>-5.1429</td>
<td>-0.1429</td>
<td>-0.5714</td>
<td>-4.0</td>
</tr>
</tbody>
</table>

Which variable must leave and which must enter?

See next page...
Answer is: $x_2$ leaves, $x_1$ enters.

Resulting dictionary is OPTIMAL:
**Dual-Based Phase I Method**

### Example:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>+</th>
<th>-</th>
<th>x1</th>
<th>+</th>
<th>-</th>
<th>x2</th>
<th>+</th>
<th>-</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>obj</td>
<td>0.0</td>
<td>+4.0</td>
<td>x1</td>
<td>+2.0</td>
<td>x2</td>
<td>+3.0</td>
<td>x3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w1</td>
<td>0.0</td>
<td>+1.0</td>
<td>x1</td>
<td>-2.0</td>
<td>x2</td>
<td>-1.0</td>
<td>x3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w2</td>
<td>0.0</td>
<td>+1.0</td>
<td>x1</td>
<td>-3.0</td>
<td>x2</td>
<td>-3.0</td>
<td>x3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w3</td>
<td>-3.0</td>
<td>+1.0</td>
<td>x1</td>
<td>-1.0</td>
<td>x2</td>
<td>-1.0</td>
<td>x3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w4</td>
<td>-1.0</td>
<td>+1.0</td>
<td>x1</td>
<td>-2.0</td>
<td>x2</td>
<td>0.0</td>
<td>x3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it’s dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we’ll use it in another algorithm later.

**Phase I—First Pivot:** $w_3$ leaves, $x_1$ enters.

After first pivot...
Dual-Based Phase I Method—Second Pivot

Current dictionary:

<table>
<thead>
<tr>
<th>obj</th>
<th>12.0</th>
<th>4.0</th>
<th>6.0</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>-6.0</td>
<td>3.0</td>
<td>-1.0</td>
<td>3.0</td>
<td>-2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>w2</td>
<td>-9.0</td>
<td>4.0</td>
<td>-3.0</td>
<td>3.0</td>
<td>-6.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>x1</td>
<td>3.0</td>
<td>-1.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>-2.0</td>
</tr>
<tr>
<td>x2</td>
<td>5.0</td>
<td>-1.0</td>
<td>-2.0</td>
<td>-2.0</td>
<td>2.0</td>
<td>-2.0</td>
</tr>
</tbody>
</table>

Dual pivot: \( w_2 \) leaves, \( x_2 \) enters.

After pivot:

<table>
<thead>
<tr>
<th>obj</th>
<th>3.0</th>
<th>1.0</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>-1.5</td>
<td>1.0</td>
<td>-1.0</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>x2</td>
<td>1.5</td>
<td>-0.6667</td>
<td>-0.5</td>
<td>-0.1667</td>
<td>0.1667</td>
</tr>
<tr>
<td>x1</td>
<td>1.5</td>
<td>-0.3333</td>
<td>-0.5</td>
<td>0.1667</td>
<td>-1.1667</td>
</tr>
<tr>
<td>x4</td>
<td>2.0</td>
<td>0.3333</td>
<td>-1.0</td>
<td>0.3333</td>
<td>-2.3333</td>
</tr>
</tbody>
</table>
Current dictionary:

Dual pivot:
\( w_1 \) leaves,
\( w_2 \) enters.

After pivot:

It’s feasible!
Current dictionary:

It’s feasible.

Ignore fake objective.

Use the real thing (top row).

Primal pivot: $x_3$ enters, $w_4$ leaves.
After pivot:

Problem is **unbounded!**