Rock-Paper-Scissors

A two person game.

Rules: At the count of three declare one of:

Rock    Paper    Scissors

Winner Selection. Identical selection is a draw. Otherwise:

- Rock “dulls” Scissors  ... in other words ...  Rock beats Scissors
- Scissors “cuts” Paper   Scissors beats Paper
- Paper “covers” Rock     Paper beats Rock

Check out Sam Kass’ version: Rock, Paper, Scissors, Lizard, Spock

It was featured on an episode of The Big Bang Theory.
Payoff Matrix

Payoffs are *from* row player *to* column player:

\[
A = \begin{bmatrix}
R & P & S \\
R & 0 & 1 & -1 \\
P & -1 & 0 & 1 \\
S & 1 & -1 & 0
\end{bmatrix}
\]

*Note:* Any deterministic strategy employed by either player can be defeated systematically by the other player.
Given: $m \times n$ matrix $A$.

- **Row player** selects a **strategy** $i \in \{1, \ldots, m\}$.
- **Col player** selects a **strategy** $j \in \{1, \ldots, n\}$.
- Row player pays col player $a_{ij}$ dollars.

**Note:** The rows of $A$ represent deterministic strategies for the row player, while columns of $A$ represent deterministic strategies for the col player.

**Deterministic strategies are usually bad.**
Randomized Strategies

- Suppose row player picks $i$ with probability $y_i$.
- Suppose col player picks $j$ with probability $x_j$.

Throughout, $x = [x_1 \ x_2 \ \cdots \ x_n]^T$ and $y = [y_1 \ y_2 \ \cdots \ y_m]^T$ will denote stochastic vectors:

$$x_j \geq 0, \quad j = 1, 2, \ldots, n$$

$$\sum_j x_j = 1.$$  

If row player uses random strategy $y$ and col player uses $x$, then expected payoff from row player to col player is

$$\sum_i \sum_j y_i a_{ij} x_j = y^T Ax.$$
Column Player’s Analysis

Suppose col player were to adopt strategy $x$.

Then, row player’s best defense is to use $y$ that minimizes the expected payment:

$$\min_y y^T A x$$

And so col player should choose that $x$ which maximizes these possibilities:

$$\max_x \min_y y^T A x$$
Solving Max-Min Problems as LPs

Inner optimization is easy:

$$\min_y y^T Ax = \min_i e_i^T Ax$$

($e_i$ denotes the vector that's all zeros except for a one in the $i$-th position—that is, deterministic strategy $i$).

**Note:** Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

$$\max (\min_i e_i^T Ax)$$

$$\sum_j x_j = 1,$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n.$$
Introduce a scalar variable \( v \) representing the value of the inner minimization:

\[
\begin{align*}
\text{max} \ v \\
v & \leq e_i^T A x, \quad i = 1, 2, \ldots, m, \\
\sum_j x_j & = 1, \\
x_j & \geq 0, \quad j = 1, 2, \ldots, n.
\end{align*}
\]

Writing in pure matrix-vector notation:

\[
\begin{align*}
\text{max} \ v & \\
ve - Ax & \leq 0 \\
e^T x & = 1 \\
x & \geq 0
\end{align*}
\]

(\( e \) denotes the vector of all ones).
Finally, in Block Matrix Form

\[
\begin{bmatrix}
-A & e \\
e^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
\[
x \geq 0
\]
\[
v \text{ free}
\]
Similarly, row player seeks $y^*$ attaining:

$$\min_y \max_x y^T A x$$

which is equivalent to:

$$\begin{align*}
\min u \\
ue - A^T y & \geq 0 \\
e^T y &= 1 \\
y & \geq 0
\end{align*}$$
Row Player’s Problem in Block-Matrix Form

\[
\begin{align*}
\min & \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix} \\
\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} & \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
y & \geq 0 \\
u & \text{ free}
\end{align*}
\]

Note: Row player’s problem is dual to col player’s.
MiniMax Theorem

Theorem:

Let $x^*$ denote col player’s solution to her max–min problem.
Let $y^*$ denote row player’s solution to his min–max problem.
Then

$$\max_x y^*^T A x = \min_y y^T A x^*. $$

Proof. From *Strong Duality Theorem*, we have

$$u^* = v^*. $$

Also,

$$v^* = \min_i e_i^T A x^* = \min_y y^T A x^* $$

$$u^* = \max_j y^*^T A e_j = \max_x y^*^T A x$$

QED

“As far as I can see, there could be no theory of games...without that theorem...I thought there was nothing worth publishing until the Minimax Theorem was proved” – John von Neumann
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
data;
set ROWS := P S R;
set COLS := P S R;
param A: P S R:=
  P 0 1 -2
  S -3 0 4
  R 5 -6 0
;

solve;
printf {j in COLS}: " %3s %10.7f \n", j, 102*x[j];
printf {i in ROWS}: " %3s %10.7f \n", i, 102*ineqs[i];
printf: "Value = %10.7f \n", 102*v;
AMPL Output

```
ampl gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective -0.1568627451
dual objective -0.1568627451
  P  40.0000000
  S  36.0000000
  R  26.0000000
  P  62.0000000
  S  27.0000000
  R  13.0000000
Value = -16.0000000
```
Consider:

\[
\begin{align*}
    \text{max } c^T x \\
    Ax &= b \\
    x &\geq 0
\end{align*}
\]

Rewrite equality constraints as pairs of inequalities:

\[
\begin{align*}
    \text{max } c^T x \\
    Ax &\leq b \\
    -Ax &\leq -b \\
    x &\geq 0
\end{align*}
\]

Put into block-matrix form:

\[
\begin{align*}
    \text{max } c^T x \\
    \begin{bmatrix} A & -A \end{bmatrix} x &\leq \begin{bmatrix} b \\ -b \end{bmatrix} \\
    x &\geq 0
\end{align*}
\]

Dual is:

\[
\begin{align*}
    \text{min } \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \\
    \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} &\geq c \\
    y^+, y^- &\geq 0
\end{align*}
\]

Which is equivalent to:

\[
\begin{align*}
    \text{min } b^T (y^+ - y^-) \\
    A^T (y^+ - y^-) &\geq c \\
    y^+, y^- &\geq 0
\end{align*}
\]

Finally, letting \( y = y^+ - y^- \), we get

\[
\begin{align*}
    \text{min } b^T y \\
    A^T y &\geq c \\
    y &\text{ free.}
\end{align*}
\]
• Equality constraints $\implies$ free variables in dual.
• Inequality constraints $\implies$ nonnegative variables in dual.

**Corollary:**

• Free variables $\implies$ equality constraints in dual.
• Nonnegative variables $\implies$ inequality constraints in dual.
A Real-World Example
The Ultra-Conservative Investor

Consider again the historical investment data \((S_j(t))\):

![Graph showing historical investment data for various sectors.]

We can let \( R_{j,t} = S_j(t)/S_j(t-1) \) and view \( R \) as a payoff matrix in a game between Fate and the Investor.
The columns represent pure strategies for our conservative investor. The rows represent how history might repeat itself. Of course, for tomorrow, Fate won’t just repeat a previous year but, rather, will present some mixture of these previous years. Likewise, the investor won’t put all of her money into one asset. Instead she will put a certain fraction into each.

Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

**Investor’s Optimal Asset Mix:**
- XLP  90.7
- QQQQ  9.3

**Mean, old Fate’s Mix:**
- 2008-10-08  37.6
- 2008-11-28  62.4

The value of the game is the investor’s expected return, 94.3%, which is actually a loss of 5.7%.
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}: sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal: sum{j in COLS} x[j] = 1;

data;

set COLS := xlu xlb xli xlv xlf xle mdy xlk xly xlp qqqq spy;
set ROWS := include 'dates.out';

param A: xlu xlb xli xlv xlf xle mdy xlk xly xlp qqqq spy:=
include 'amplreturn3.data' ;

solve;

printf "Investor’s strategy\n";
printf {j in COLS: x[j]>0.0005}: " %40s %5.1f \n", j, 100*x[j];
printf "\n";
printf "God’s strategy\n";
printf {i in ROWS: ineqs[i]>0.0005}: " %40s %5.1f \n", i, 100*ineqs[i];