Linear Programming: Chapter 5
Duality

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Recall the resource allocation problem \((m = 2, n = 3)\):

\[
\text{maximize} \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to} \quad a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \leq b_1 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \leq b_2 \\
x_1, x_2, x_3 \geq 0,
\]

where

\[
c_j = \text{profit per unit of product } j \text{ produced} \\
b_i = \text{units of raw material } i \text{ on hand} \\
a_{ij} = \text{units raw material } i \text{ required to produce 1 unit of prod } j.
\]
Closing Up Shop

If we produce one unit less of product \( j \), then we free up:

- \( a_{1j} \) units of raw material 1 and
- \( a_{2j} \) units of raw material 2.

Selling these unused raw materials for \( y_1 \) and \( y_2 \) dollars/unit yields \( a_{1j}y_1 + a_{2j}y_2 \) dollars.

Only interested if this exceeds lost profit on each product \( j \):

\[
a_{1j}y_1 + a_{2j}y_2 \geq c_j, \quad j = 1, 2, 3.
\]

Consider a buyer offering to purchase our entire inventory. Subject to above constraints, buyer wants to minimize cost:

\[
\begin{align*}
\text{minimize} & \quad b_1y_1 + b_2y_2 \\
\text{subject to} & \quad a_{11}y_1 + a_{21}y_2 \geq c_1 \\
& \quad a_{12}y_1 + a_{22}y_2 \geq c_2 \\
& \quad a_{13}y_1 + a_{23}y_2 \geq c_3 \\
& \quad y_1, y_2 \geq 0.
\end{align*}
\]
Duality

Every Problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \\
& \quad x_j \geq 0 \quad j = 1, 2, \ldots, n,
\end{align*}
\]

Has a Dual:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n \\
& \quad y_i \geq 0 \quad i = 1, 2, \ldots, m.
\end{align*}
\]
Dual of Dual

Primal Problem:

maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \]
\[ x_j \geq 0 \quad j = 1, \ldots, n, \]

Dual in “Standard” Form:

−maximize \[ \sum_{i=1}^{m} -b_i y_i \]
subject to \[ \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j \quad j = 1, \ldots, n \]
\[ y_i \geq 0 \quad i = 1, \ldots, m. \]

Original problem is called the *primal problem.*

A problem is defined by its data (notation used for the variables is arbitrary).

Dual is “negative transpose” of primal.

**Theorem** Dual of dual is primal.
Weak Duality Theorem

If \((x_1, x_2, \ldots, x_n)\) is feasible for the primal and \((y_1, y_2, \ldots, y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i.
\]

Proof.

\[
\sum_j c_j x_j \leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j
\]

\[
= \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i
\]

\[
\leq \sum_i b_i y_i.
\]
Gap or No Gap?

An important question:

Is there a gap between the largest primal value and the smallest dual value?

Answer is provided by the Strong Duality Theorem (coming later).
Simplex Method and Duality

A Primal Problem:

<table>
<thead>
<tr>
<th>obj</th>
<th>0</th>
<th>+</th>
<th>-3</th>
<th>x1</th>
<th>+</th>
<th>2</th>
<th>x2</th>
<th>+</th>
<th>1</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>0</td>
<td></td>
<td>0</td>
<td>x1</td>
<td>-</td>
<td>-1</td>
<td>x2</td>
<td>-</td>
<td>2</td>
<td>x3</td>
</tr>
<tr>
<td>w2</td>
<td>3</td>
<td></td>
<td>-3</td>
<td>x1</td>
<td>-</td>
<td>4</td>
<td>x2</td>
<td>-</td>
<td>1</td>
<td>x3</td>
</tr>
</tbody>
</table>

Its Dual:

<table>
<thead>
<tr>
<th>obj</th>
<th>0</th>
<th>+</th>
<th>0</th>
<th>y1</th>
<th>+</th>
<th>-3</th>
<th>y2</th>
</tr>
</thead>
<tbody>
<tr>
<td>z1</td>
<td>3</td>
<td></td>
<td>0</td>
<td>y1</td>
<td></td>
<td>3</td>
<td>y2</td>
</tr>
<tr>
<td>z2</td>
<td>-2</td>
<td></td>
<td>1</td>
<td>y1</td>
<td>-</td>
<td>4</td>
<td>y2</td>
</tr>
<tr>
<td>z3</td>
<td>-1</td>
<td></td>
<td>-2</td>
<td>y1</td>
<td>-</td>
<td>1</td>
<td>y2</td>
</tr>
</tbody>
</table>

Notes:
- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: \( x_2 \) enters, \( w_2 \) leaves.
Make analogous pivot in dual: \( z_2 \) leaves, \( y_2 \) enters.
Second Iteration

After First Pivot:

Primal (feasible):

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>obj</td>
<td>3/2</td>
<td>-3/2</td>
<td>x1</td>
<td>1/2</td>
</tr>
<tr>
<td>w1</td>
<td>3/4</td>
<td>-3/4</td>
<td>x1</td>
<td>1/4</td>
</tr>
<tr>
<td>x2</td>
<td>3/4</td>
<td>-3/4</td>
<td>x1</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Dual (still not feasible):

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>obj</td>
<td>-3/2</td>
<td>-3/4</td>
<td>y1</td>
<td>-3/4</td>
</tr>
<tr>
<td>x1</td>
<td>3/2</td>
<td>3/4</td>
<td>y1</td>
<td>3/4</td>
</tr>
<tr>
<td>y2</td>
<td>1/2</td>
<td>-1/4</td>
<td>y1</td>
<td>-1/4</td>
</tr>
<tr>
<td>z3</td>
<td>-1/2</td>
<td>-9/4</td>
<td>y1</td>
<td>-1/4</td>
</tr>
</tbody>
</table>

Note: negative transpose property intact.

Again, use primal to pick pivot: \(x_3\) enters, \(w_1\) leaves.

Make analogous pivot in dual: \(z_3\) leaves, \(y_1\) enters.
After Second Iteration

Primal:

- Is optimal.

Dual:

- Negative transpose property remains intact.
- Is optimal.

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.
Strong Duality Theorem

Conclusion on previous slide is the essence of the **strong duality theorem** which we now state:

**Theorem.** *If the primal problem has an optimal solution,*

\[ x^* = (x_1^*, x_2^*, \ldots, x_n^*), \]

*then the dual also has an optimal solution,*

\[ y^* = (y_1^*, y_2^*, \ldots, y_m^*), \]

*and*

\[ \sum_j c_j x_j^* = \sum_i b_i y_i^*. \]

**Paraphrase:**

If primal has an optimal solution, then there is no duality gap.
Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

Example of infinite gap:

\[
\begin{align*}
\text{maximize} & \quad 2x_1 - x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad -x_1 + x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
Theorem. At optimality, we have

\[ x_j z_j = 0, \quad \text{for} \ j = 1, 2, \ldots, n, \]
\[ w_i y_i = 0, \quad \text{for} \ i = 1, 2, \ldots, m. \]
Proof

Recall the proof of the Weak Duality Theorem:

\[
\sum_j c_j x_j \leq \sum_j (c_j + z_j) x_j = \sum_j \left( \sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j
\]

\[
= \sum_i \left( \sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \leq \sum_i b_i y_i,
\]

The inequalities come from the fact that

\[x_j z_j \geq 0, \quad \text{for all } j,\]

\[w_i y_i \geq 0, \quad \text{for all } i.\]

By Strong Duality Theorem, the inequalities are equalities at optimality.
Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

An Example. Showing both primal and dual dictionaries:

Looking at dual dictionary: $y_2$ enters, $z_2$ leaves.

On the primal dictionary: $w_2$ leaves, $x_2$ enters.

After pivot...
Dual Simplex Method: Second Pivot

Going in, we have:

Looking at dual:

$y_1$ enters, $z_4$ leaves.

Looking at primal:

$w_1$ leaves, $x_4$ enters.
Dual Simplex Method Pivot Rule

Refering to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...
Going in, we have:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>x4</th>
<th>x2</th>
<th>w3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-15.7143</td>
<td>-5.1429</td>
<td>-0.1429</td>
</tr>
<tr>
<td>+</td>
<td></td>
<td>-5.1429</td>
<td>x1</td>
<td>-0.1429</td>
</tr>
<tr>
<td>-</td>
<td></td>
<td>-0.1429</td>
<td>x1</td>
<td>-0.5714</td>
</tr>
<tr>
<td>-</td>
<td></td>
<td>-0.2857</td>
<td>w2</td>
<td>-0.1429</td>
</tr>
<tr>
<td>-</td>
<td></td>
<td>-0.2857</td>
<td>w2</td>
<td>0.0</td>
</tr>
<tr>
<td>+</td>
<td></td>
<td>0.0</td>
<td>x3</td>
<td>0.0</td>
</tr>
<tr>
<td>+</td>
<td></td>
<td>0.0</td>
<td>x3</td>
<td>0.0</td>
</tr>
<tr>
<td>+</td>
<td></td>
<td>-1.4286</td>
<td>w2</td>
<td>0.2857</td>
</tr>
</tbody>
</table>

Which variable must leave and which must enter?

See next page...
Answer is: $x_2$ leaves, $x_1$ enters.

Resulting dictionary is OPTIMAL:

\[
\begin{array}{cccccccc}
\text{obj} & = & -17.0 & + & -9.0 & x_2 & + & -1.0 & w_2 & + & 0.0 & x_3 & + & -4.0 & v_1 \\
\text{x}_4 & = & 2.75 & - & -0.25 & x_2 & - & -0.25 & v_2 & - & 0.0 & x_3 & - & -0.5 & v_1 \\
\text{x}_1 & = & 0.25 & - & -1.75 & x_2 & - & 0.25 & v_2 & - & 0.0 & x_3 & - & -0.5 & v_1 \\
\text{w}_3 & = & 2.0 & - & 7.0 & x_2 & - & 0.0 & v_2 & - & 3.0 & x_3 & - & 2.0 & v_1 \\
\end{array}
\]
Dual-Based Phase I Method

Example:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>+</th>
<th>-</th>
<th>+</th>
<th>-</th>
<th>+</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>2.0</td>
<td>1.0</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>w2</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>3.0</td>
<td>1.0</td>
<td>4.0</td>
<td>1.0</td>
</tr>
<tr>
<td>w3</td>
<td>-3.0</td>
<td>1.0</td>
<td>0.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>w4</td>
<td>-1.0</td>
<td>1.0</td>
<td>0.0</td>
<td>-2.0</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For Phase I, use the fake objective—it’s dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we’ll use it in another algorithm later.

Phase I—First Pivot: $w_3$ leaves, $x_1$ enters.
After first pivot...
Dual-Based Phase I Method—Second Pivot

Recall current dictionary:

Dual pivot: $w_2$ leaves, $x_2$ enters.

After pivot:
Current dictionary:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.0</td>
<td>-1.5</td>
<td>1.0</td>
<td>-1.0</td>
<td>0.5</td>
<td>-0.5</td>
<td>5.5</td>
<td>-2.0</td>
</tr>
</tbody>
</table>

Dual pivot:

- $w_1$ leaves,
- $w_2$ enters.

After pivot:

It’s feasible!
Current dictionary:

It’s feasible.

Ignore fake objective.

Use the real thing (top row).

Primal pivot: $x_3$ enters, $w_4$ leaves.
## Final Dictionary

After pivot:

<table>
<thead>
<tr>
<th></th>
<th>obj</th>
<th>6.75</th>
<th>+</th>
<th>4.5</th>
<th>w3</th>
<th>+</th>
<th>-2.0</th>
<th>w3</th>
<th>+</th>
<th>1.0</th>
<th>w1</th>
<th>+</th>
<th>-6.75</th>
<th>v4</th>
</tr>
</thead>
<tbody>
<tr>
<td>w2</td>
<td>11.25</td>
<td>+</td>
<td>6.25</td>
<td>-</td>
<td>-6.5</td>
<td>v3</td>
<td>-</td>
<td>3.5</td>
<td>v1</td>
<td>-</td>
<td>8.25</td>
<td>v4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x2</td>
<td>3.25</td>
<td>+</td>
<td>0.25</td>
<td>-</td>
<td>-1.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
<td>v1</td>
<td>-</td>
<td>1.25</td>
<td>v4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x1</td>
<td>0.5</td>
<td>+</td>
<td>-0.5</td>
<td>-</td>
<td>0.0</td>
<td>v3</td>
<td>-</td>
<td>0.0</td>
<td>v1</td>
<td>-</td>
<td>-0.5</td>
<td>v4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x3</td>
<td>0.75</td>
<td>+</td>
<td>0.75</td>
<td>-</td>
<td>-0.5</td>
<td>v3</td>
<td>-</td>
<td>0.5</td>
<td>v1</td>
<td>-</td>
<td>0.75</td>
<td>v4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem is **unbounded!**