Problem 4.33 (Solution by Dongning Guo, June 20, 2001)

(a) The near far resistance of user 1 can be obtained as

\[ \eta_1 = \min_{a_2, a_3} \| s_1 + a_2 s_2 + a_3 s_3 \|^2 \]

\[ = \min_{a_2, a_3} (1 + a_2^2 + a_3^2 + 2a_2 \rho_{12} + 2a_3 \rho_{13} + 2a_2 a_3 \rho_{23}). \]

It is a quadratic form in \( a_2 \) and \( a_3 \). By letting the differentiation with respect to \( a_2 \) and \( a_3 \) be 0, we have that the worst case coefficients satisfy that

\[ \begin{cases} a_2 + a_3 \rho_{23} = -\rho_{12}, \\ \rho_{23} a_2 + a_3 = -\rho_{13}. \end{cases} \]

Obviously, the minimum is achieved at

\[ \begin{cases} \hat{a}_2 = (\rho_{23} \rho_{13} - \rho_{12})/(1 - \rho_{23}^2), \\ \hat{a}_3 = (\rho_{23} \rho_{12} - \rho_{13})/(1 - \rho_{23}^2). \end{cases} \]

With some diligence, we have

\[ \eta_1 = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2 \rho_{12} \rho_{13} \rho_{23} \]

(b) \( \eta_1(\alpha_3) \) is seeking the minimum of the same energy expression as that of the unconstrained minimum \( \eta_1 \) in a constrained region of

\[ |a_3| \leq \alpha_3. \]

If \( \alpha_3 \geq |\hat{a}_3| \), then the unconstrained minimum is achieved within the constrained region. Hence \( \eta_1(\alpha_3) = \eta_1 \). Otherwise, the minimum is achieved on the border of the constrained region, i.e., \( a_3 = \pm \alpha_3 \), so that the minimum is sought as

\[ \eta_1(\alpha_3) = \min_{a_2} \left( 1 + a_2^2 + \alpha_3^2 + 2a_2 \rho_{12} \pm 2 \alpha_3 (\rho_{13} + a_2 \rho_{23}) \right). \]

Clearly, at the minimum, \( a_2 \) satisfies \( a_2 = -\rho_{12} \mp \alpha_3 \rho_{23} \). We choose the sign that leads to the minimum. In all,

\[ \eta_1(\alpha_3) = \begin{cases} \eta_1, & \text{if } \alpha_3 \geq |\rho_{23} \rho_{12} - \rho_{13}| \frac{1}{1 - \rho_{23}^2}, \\ 1 - \rho_{12}^2 - 2|\rho_{23} \rho_{12} - \rho_{13}| \alpha_3 + (1 - \rho_{23}^2) \alpha_3^2, & \text{otherwise}. \end{cases} \]