Problem 4.45

Show that for any set of K signature waveforms and offsets, the asynchronous minimum distance \( d_{k,\text{min}} \) is always achieved by an error sequence with \( w(\bar{c}) \leq 2KE^{K-1} \).

Proposed solution

Assume each user transmits \( 2*M+1 \) bits. The error vector\(^{1}\) \( \bar{c} \) shall have \((2*M+1)K\) dimensions, i.e. it shall lie in the \( \{-1,0,1\}^{(2*M+1)K} \) space. Let us instead represent \( \bar{c} \) as a \((2*M+1)\)-dimensional vector, where each entry is a vector in the \( \{-1,0,1\}^{K} \) space: \( \bar{c} = [\bar{v}_{1}, \bar{v}_{2}, ..., \bar{v}_{K}] \)

\[
\|S(\bar{c})\|^{2} = \bar{c}^{T}H\bar{c}
\]

where, from 4.26:

\[
H = A_{M}R_{A}A_{M} = \begin{pmatrix}
H[0] & H^{T}[1] & 0 & \ldots & 0 & 0 \\
H[1] & H[0] & H^{T}[1] & \ldots & 0 & 0 \\
0 & H[1] & H[0] & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & H[0] & H^{T}[1] \\
\end{pmatrix}
\]

\[
\]

Hence:

\[
\|S(\bar{c})\|^{2} = \sum_{k=-M}^{M-1} [\bar{v}_{k}^{T}H[0]\bar{v}_{k} + 2\bar{v}_{k+1}^{T}H[1]\bar{v}_{k}] + \bar{v}_{K}^{T}H[0]\bar{v}_{K} \tag{1}
\]

It is readily observable that if \( \bar{c} \) is to be a minimum distance vector of least weight, if \( v_{j} = 0 \) then \( v_{i} = 0 \) either for \( i < j \) or for \( i > j \) (but obviously not both because \( \bar{c} \neq 0 \)). The reason for this is that we would be able to split the sum in (1) in two sums:

\[
\sum_{k=-M}^{j-2} [\bar{v}_{k}^{T}H[0]\bar{v}_{k} + 2\bar{v}_{k+1}^{T}H[1]\bar{v}_{k}] + \bar{v}_{j-1}^{T}H[0]\bar{v}_{j-1}
\]

\footnote{vector’ and ‘column vector’ are synonymous throughout this solution; ‘left’ and ‘right’ may be used to imply the traditional left-to-right low-index to high-index vector notation}
\[ \sum_{m=j+1}^{M-1} \left[ \vec{v}_k^T \mathbf{H}[0] \vec{v}_k + 2 \vec{v}_{k+1}^T \mathbf{H}[1] \vec{v}_k \right] + \vec{v}_K^T \mathbf{H}[0] \vec{v}_K \]

both summing to non-negative values (because \( \mathbf{R} \) – and hence \( \mathbf{H} \) – is non-negative definite). One of those sums corresponds to a valid error vector of weight strictly less than that of \( \vec{e} \). The other one would not correspond to a valid error vector because it would ‘zero’ out the entry for the bit of interest.

We have established that a minimum distance error of least weight should be composed of a contiguous string of non-zero \( \vec{v}_j \) K-vectors, possibly surrounded by zero K-vectors. Let us assume that \( \vec{v}_j \neq \vec{0} \) for \( M_1 \leq j \leq M_2 \) and \( \vec{v}_j = \vec{0} \) outside of this range. We then have:

\[
\|S(\vec{e})\|^2 = \sum_{k=M_1}^{M_2-1} \left[ \vec{v}_k^T \mathbf{H}[0] \vec{v}_k + 2 \vec{v}_{k+1}^T \mathbf{H}[1] \vec{v}_k \right] + \vec{v}_{M_2}^T \mathbf{H}[0] \vec{v}_{M_2} \tag{2}
\]

In what follows we’ll ignore the restriction that we need to find the minimum error distance for a particular bit \([\text{of a particular user}]\); we’ll add this restriction at the end of our analysis and adjust the result there.

We make the following observations concerning the minimum distance, least weight vector \( \vec{e} \):

**Observation 1:** We now show that there cannot be any two indices \( m < n \) for which \( \vec{v}_m = \vec{v}_n \neq \vec{0} \). If two such indices existed, it must be that

\[
\sum_{k=m}^{n-1} \left[ \vec{v}_k^T \mathbf{H}[0] \vec{v}_k + 2 \vec{v}_{k+1}^T \mathbf{H}[1] \vec{v}_k \right] < 0
\]

for otherwise the vector \( \vec{e}_n = [\vec{0}, \ldots, \vec{0}, \vec{v}_{M_1}, \ldots, \vec{v}_m, \vec{v}_{n+1}, \ldots] \) would achieve at least as small a distance with less weight. This would contradict the assumption that \( \vec{e} \) was optimal. If the sum give above was indeed negative, in the case of sufficiently large \( M \), we would be able to form a vector repeating the sequence \([\vec{v}_m, \ldots, \vec{v}_{n-1}]\) sufficiently many times (and eventually ending in \( \vec{v}_m \)) to make (2) negative. This would contradict the non-negative definiteness of \( \mathbf{H} \).

**QED**

**Observation 2:** We next show that there cannot be any two indices \( m < n \) for which \( \vec{v}_m = -\vec{v}_n \neq \vec{0} \). Let us define:

\[
\beta = \sum_{k=m}^{n-1} \left[ \vec{v}_k^T \mathbf{H}[0] \vec{v}_k + 2 \vec{v}_{k+1}^T \mathbf{H}[1] \vec{v}_k \right] - \vec{v}_m^T \mathbf{H}[0] \vec{v}_m
\]
If two such indices \( m, n \) existed, it must be that

\[
\sum_{k=m}^{n-1} [\vec{v}^T_k \mathbf{H}[0] \vec{v}_k + 2 \vec{v}^T_{k+1} \mathbf{H}[1] \vec{v}_k] = \beta + \vec{v}^T_m \mathbf{H}[0] \vec{v}_m = \beta + \vec{v}^T_n \mathbf{H}[0] \vec{v}_n > 0 \quad (3)
\]

This is so because we can construct a sequence:

\[
[\vec{v}_n, \vec{v}_{n-1}, \ldots, \vec{v}_m, \vec{v}_{m+1}, \ldots, \vec{v}_{n-1}, \vec{v}_n] = [\vec{u}_1, \ldots, \vec{u}_{2(n-m)+1}]
\]

for which \( \vec{u}_1 = \vec{u}_{2(n-m)+1} = \vec{v}_m \). From the argument in \textbf{observation 1}, we know that

\[
2 \sum_{k=1}^{2(n-m)} [\vec{u}^T_k \mathbf{H}[0] \vec{u}_k + 2 \vec{u}^T_{k+1} \mathbf{H}[1] \vec{u}_k] \geq 0
\]

which implies \( 2(\beta + \vec{v}^T_n \mathbf{H}[0] \vec{v}_n) > 0 \), from which (3) results. But (3) essentially says that if we negate the vectors \( \vec{v}_{M1} \ldots \vec{v}_m \) and perform a shift similar to that in \textbf{observation 1}, we obtain a new vector \( \vec{\epsilon}_n \) with equal or less distance and lesser weight. Again, this contradicts the optimality in the choice of \( \vec{\epsilon} \). QED

There are \( 3^K \) different vectors \( \vec{v}_j \). From \textbf{observation 1} \( \vec{\epsilon} \) can contain at most \( 3^K \) K-vectors. This number is reduced to \( \frac{3^K}{2} \) by \textbf{observation 2}. Moreover, there are K elements of \( \{-1, 0, 1\} \) in each of the K-vectors, for a total of \( \frac{K3^K}{2} \) elements. In the worst case (if all K-vector choices unique up to negation are used) a third of these elements are zeroes. There are therefore at most \( (1 - \frac{1}{3}) \frac{K3^K}{2} = K3^{K-1} \) non-zero elements (i.e. the weight is at most \( K3^{K-1} \)). While this looks ‘better’ by a factor of 2 than what the problem requires, we soon realize we need to cope with the restriction that the minimum distance is sought for a particular bit of a particular user.

The situation is easily patched by noting that observations 1 and 2 are applicable separately to contiguous K-vector ordered sets to the left and to the right of the K-vector that contains the desired bit, so we must multiply our upper bound by 2. Hence, no matter how large \( M \) is, a minimum distance can always be found for an error vector \( \vec{\epsilon} \) having weight upper bounded by \( 2K3^{K-1} \).