Thus, we can get average bit error rate for user 1:

\[
P_1(A_2, \sigma) = 0.25P_1(A_2, \sigma | b_1 = 1, b_2 = 1) + 0.25P_1(A_2, \sigma | b_1 = 1, b_2 = 0) + 0.25P_1(A_2, \sigma | b_1 = 0, b_2 = 1) + 0.25P_1(A_2, \sigma | b_1 = 0, b_2 = 0)
\]

\[
= \frac{1}{2}Q \left( \frac{1 + 2A_2}{2\sigma} \sqrt{\int_0^T s^4(t)dt} \right) + \frac{1}{2}Q \left( \frac{1}{2\sigma} \sqrt{\int_0^T s^4(t)dt} \right)
\]

(25)

Thus:  

\[
P_1(0, \sigma) = Q \left( \frac{1}{2\sigma} \sqrt{\int_0^T s^4(t)dt} \right)
\]

(26)

Finally, we have:  

\[
\lim_{A_2 \to \infty} \frac{P_1(A_2, \sigma)}{P_1(0, \sigma)} = \frac{1}{2} \text{ by the property of Q-function}
\]

3 Problem 4.58

Solution: Before we go on to part (a), we first explore the property of \( \langle S(\epsilon'), S(\epsilon'') \rangle \). In this case, the correlation matrix is now (by the equivalence of SNR):

\[
R = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1K} \\
\rho_{12} & 1 & 0 & \cdots & 0 \\
\rho_{13} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{1K} & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

(27)

We then have:

\[
H = ARA = A^2R
\]

(28)

Therefore, if \( \epsilon' + \epsilon'' \) is a decomposition of some error vector \( \epsilon \), then

\[
\sum_{k=1}^{K} \epsilon'_k \epsilon''_k = 0 \quad \text{since corresponding components cannot be both nonzero}
\]

(29)
\( \langle S(\epsilon'), S(\epsilon'') \rangle = \left( \sum_{k=1}^{K} \rho_{1k} \epsilon'_k, \rho_{12} \epsilon'_1 + \epsilon'_2, \cdots, \rho_{1K} \epsilon'_1 + \epsilon'_K \right) \left( \begin{array}{c} \epsilon''_1 \\ \epsilon''_2 \\ \vdots \\ \epsilon''_K \end{array} \right) \)

\[ = \epsilon''_1 \sum_{k=1}^{K} \rho_{1k} \epsilon'_k + \sum_{k=2}^{K} \epsilon''_k (\rho_{1k} \epsilon'_1 + \epsilon'_k) \]

\[ = \sum_{k=2, \epsilon_k \neq 0}^{K} (\epsilon''_k \epsilon'_k + \epsilon'_1 \epsilon''_k) \quad (30) \]

Then we do some discussion about each term of this inner product, so that we can get some insight into the property of the nonnegative definiteness of the inner product.

\[ \forall k \geq 2, \text{ each term in the sum is as follows:} \]

\[ \epsilon''_k = 0 \Rightarrow \rho_{1k} \epsilon''_1 \epsilon'_k \quad (31) \]

\[ \epsilon''_k = 1 \Rightarrow \rho_{1k} \epsilon'_1 \quad (32) \]

\[ \epsilon''_k = -1 \Rightarrow -\rho_{1k} \epsilon'_1 \quad (33) \]

We now turn to the problem itself.

### 3.1 (a)

Since \( \langle S(\epsilon'), S(\epsilon'') \rangle = \langle S(\epsilon''), S(\epsilon') \rangle \), when we talk about \( F_1 \), we specify \( \epsilon'_1 = 1, \epsilon''_1 = 0 \) or \( \epsilon'_1 = -1, \epsilon''_1 = 0 \). When \( \epsilon'_1 = 1, \epsilon''_1 = 0 \), since \( \epsilon'_k, \epsilon''_k \) cannot be both zero, we consider the 'best case' that we try to decompose the error vector as best as we can. The main criterion here for the error vector to be decomposable is whether the inner product (30) is nonnegative definite or not. But in this criterion, 0 component in the (original) error vector has no effect since both the \( \epsilon', \epsilon'' \) are zero and has no effect on the inner product.

Note again that to discuss \( F_1 \), the inner product (30) becomes:

\[ \sum_{k=2, \epsilon_k \neq 0}^{K} \rho_{1k} \epsilon''_k \quad (34) \]

If the error vector \( \epsilon \) has a component \( \epsilon_k = 1 \), where \( k \geq 2 \). Since the inner product (34) has only terms about \( \epsilon''_k \), we know we can always let \( \epsilon''_k = 1, \epsilon'_k = 0 \), and if we have \( \epsilon_k = -1 \), we put \( \epsilon''_k = 0, \epsilon'_k = -1 \), then
we can always decompose such kind of error vector. That is to say, if we have at least 2 nonzero components (including $\epsilon_1$), then we are not allowed to have components of value 1. Conversely, if we have at least 2 nonzero components (including $\epsilon_1$), and all are -1 except $\epsilon_1$, then we must have at least one $\epsilon''_k = -1$ since $\epsilon''$ cannot be 0. Thus from (34) we know this vector is not decomposable (we've considered the best case).

The remaining case is trivial, i.e. if we only have one nonzero component, the $\epsilon_1$, it is obvious indecomposable.

In exactly the same way we can also derive that if $\epsilon_1 = -1$ and at least 2 components of $\epsilon$ are not zero (including $\epsilon_1$), then all nonzero components must be 1 except $\epsilon_1$. Also if we only have one nonzero component, i.e. $\epsilon_1 = -1$, it is indecomposable.

To summarize, we can write:

$$F_1 = \{(1, f_{12}, f_{13}, \cdots, f_{1K})^T, f_{1i} \in \{0, -1\}\} \cup \{(-1, f_{12}, f_{13}, \cdots, f_{1K})^T, f_{1i} \in \{0, 1\}\}$$

(35)

3.2 (b)

Like the discussion in part (a), we note first that 0 components in $\epsilon$ does not affect if it is decomposable. So when we talk about $F_2$, we need only to discuss about its nonzero components. We also specify that $\epsilon'_2 = 1, \epsilon''_2 = 0$ or $\epsilon'_2 = -1, \epsilon''_2 = 0$ because inner product (30) is symmetric in $\epsilon'$ and $\epsilon''$. We use the same method as we did in part (a), i.e. do our best to decompose $\epsilon$, and if we cannot decompose it in the best case, then it is indecomposable.

We divide our discussion into 5 cases, assuming first $\epsilon_2 = 1$:

(i) $\epsilon'_1 = 1, \epsilon''_1 = 0, \epsilon'_2 = 1, \epsilon''_2 = 0 (\epsilon_1 = 1)$

In this case, the first term $\rho_{12} \epsilon''_1 \epsilon'_2$ in the inner product is 0, so it has no effect. Since error vectors cannot be 0, there must be some nonzero component for $k \geq 3$. If $\epsilon$ has $\epsilon_k = 1$ for some $k \geq 3$, we can let $\epsilon''_k = 1$ and the corresponding term in the inner product is positive. If $\epsilon_k = -1$ for some $k \geq 3$, then we can let $\epsilon'_k = -1$ and the corresponding term in the inner product is zero. So under the first case, the error vector is always decomposable.

(ii) $\epsilon'_1 = 0, \epsilon''_1 = 1, \epsilon'_2 = 1, \epsilon''_2 = 0 (\epsilon_1 = 1)$

In this case we can always let $\epsilon''_k = \epsilon_k, \epsilon'_k = 0, \forall k \geq 3$, and these terms in inner product (30) are all zeros. Then we can always decompose the error vector in this case.
(iii) $\epsilon_1' = -1, \epsilon_1'' = 0, \epsilon_2' = 1, \epsilon_2'' = 0 (\epsilon_1 = -1)$
Under this case, we can always let $\epsilon_k' = 1, \epsilon_k'' = 0$ if $\epsilon_k = 1$ and let $\epsilon_k' = 0, \epsilon_k'' = -1$ if $\epsilon_k = -1$, then all terms in inner product (30) are zero or positive. Thus we can always decompose the error vector in this case.

(iv) $\epsilon_1' = 0, \epsilon_1'' = -1, \epsilon_2' = 1, \epsilon_2'' = 0 (\epsilon_1 = -1)$
We can let $\epsilon_k'' = \epsilon_k, \epsilon_k' = 0, \forall k \geq 3$, then all terms in the inner product (30) are zero. Thus the error vector is always decomposable in this case.

(v) $\epsilon_1' = 0, \epsilon_1'' = 0, \epsilon_2' = 1, \epsilon_2'' = 0 (\epsilon_1 = 0)$
If we have one additional nonzero component for $k \geq 3$, except the $\epsilon_2$, then we can let $\epsilon_k'' = \epsilon_k, \epsilon_k' = 0, \forall k \geq 3$, therefore all terms in the inner product (30) are zero, thus decomposable. Then the only case that we have an indecomposable error vector is that $\epsilon_2 = 1, \epsilon_k = 0, \forall k \neq 2$
To summarize, we have only one indecomposable error vector if $\epsilon_2 = 1$ and it is $\epsilon = (0, 1, 0, 0, \ldots, 0, 0)^T$
With very similar argument, we can show that if $\epsilon_2 = -1$, for $\epsilon$ to be indecomposable, we must have $\epsilon = (0, -1, 0, 0, \ldots, 0, 0)^T$
Thus we’ve got:
\[ F_2 = \{(0, 1, 0, 0, \ldots, 0, 0)^T, (0, -1, 0, 0, \ldots, 0, 0)^T\} \]

3.3 (c)
We first derive an expression for $\|S(\epsilon)\|^2$:

\[
\|S(\epsilon)\|^2 = \left(\sum_{k=1}^{K} \rho_{1k} \epsilon_k, \rho_{12} \epsilon_1 + \epsilon_2, \rho_{1K} \epsilon_1 + \epsilon_K \right) \begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_K
\end{pmatrix}
\]

\[
= \epsilon_1 \sum_{k=1}^{K} \rho_{1k} \epsilon_k + \sum_{k=2}^{K} \epsilon_k (\rho_{1k} \epsilon_1 + \epsilon_k)
\]

\[
= A^2 \left[ \epsilon_1^2 + \sum_{k=2}^{K} (\epsilon_k^2 + 2 \rho_{1k} \epsilon_1 \epsilon_k) \right] \tag{36}
\]
Then we obtain:

\[ d_{k, \text{min}}^2 = A^2 \epsilon \in \{-1, 0, 1\} \epsilon_k = 1 \left\{ \epsilon_1^2 + \sum_{k=2}^{K} (\epsilon_k^2 + 2 \rho_{1k} \epsilon_1 \epsilon_k) \right\} \] (37)

According to (37), we have:

\[
\begin{align*}
    d_{1, \text{min}}^2 &= A^2 \min_{\epsilon_2, \epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ 1 + \sum_{k=2}^{K} \epsilon_k^2 + 2 \rho_{1k} \epsilon_1 \epsilon_k \} \\
    &= A^2 \{ 1 + \sum_{k=2}^{K} \min\{1 - 2 \rho_{1k}, 0, 1 + 2 \rho_{1k}\} \} \\
    &= A^2 \{ 1 + \sum_{k=2}^{K} \min\{1 - 2 \rho_{1k}\} \} \\
    &= A^2 \{ 1 + \frac{1}{2} \sum_{k=2}^{K} (2 - 2 \rho_{1k} - |1 - 2 \rho_{1k}|) \} \\
    &= \frac{1}{2} A^2 \{ K + 1 - 2 \sum_{k=2}^{K} \rho_{1k} - \sum_{k=2}^{K} |1 - 2 \rho_{1k}| \} \quad \text{(38)}
\end{align*}
\]

\[
\begin{align*}
    d_{2, \text{min}}^2 &= A^2 \epsilon_1, \epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\} \epsilon_2 = 1 \left\{ \epsilon_1^2 + \sum_{k=2}^{K} (\epsilon_k^2 + 2 \rho_{1k} \epsilon_1 \epsilon_k) \right\} \\
    &= A^2 \min_{\epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ 1 + \sum_{k=3}^{K} \epsilon_k^2 + \sum_{k=3}^{K} (\epsilon_k^2 + 2 \rho_{1k} \epsilon_k) + 2 + 2 \rho_{12}, \\
    & \quad \sum_{k=3}^{K} (\epsilon_k^2 - 2 \rho_{1k} \epsilon_k) + 2 - 2 \rho_{12} \} \quad \text{by letting } \epsilon_1 = 0, 1, -1 \\
    &= A^2 \min_{\epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ 1 + \sum_{k=3}^{K} \epsilon_k^2 \}, \\
    & \quad \min_{\epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ \sum_{k=3}^{K} (\epsilon_k^2 - 2 \rho_{1k} \epsilon_k) + 2 - 2 \rho_{12} \} \\
    &= A^2 \min_{\epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ 1 + \sum_{k=3}^{K} \min\{1 - 2 \rho_{1k}, 0\} + 2 - 2 \rho_{12} \} \\
    &= A^2 \min_{\epsilon_3, \ldots, \epsilon_K \in \{-1, 0, 1\}} \{ 1 + \frac{1}{2} \sum_{k=3}^{K} (1 - 2 \rho_{1k} - |1 - 2 \rho_{1k}|) + 2 - 2 \rho_{12} \} \quad \text{(39)}
\end{align*}
\]
3.4 (d)

For the correlation matrix (27), let $D_K$ represent its determinant, then according to Laplace expansion we have:

$$D_K = D_{K-1} - \rho_{1K}^2$$

Thus:

$$D_K = 1 - \sum_{k=2}^{K} \rho_{1k}^2$$

(40)

where we assume $R$ is positive definite, which is equivalent to

$$1 - \sum_{k=2}^{K} \rho_{1k}^2 > 0$$

(41)

The optimum near-far resistance for user 1 is

$$\bar{\eta}_k = \frac{1}{(R^{-1})_{kk}}$$

$$= \frac{1}{\text{adj}(R)_{kk}}$$

$$= \frac{1}{\det R}$$

$$= \det R$$

$$= 1 - \sum_{k=2}^{K} \rho_{1k}^2$$

(42)

3.5 (e)

Jointly optimal decision rule for users 2, 3, $\cdots$, $K$ is

$$(b_2, b_3, \cdots, b_K)^T = \arg \max_{(b_2, b_3, \cdots, b_K)^T \in \{-1, 1\}^{K-1}} \{\Omega(1, b_2, b_3, \cdots, b_K) + \Omega(-1, b_2, b_3, \cdots, b_K)\}$$

(43)

where $\Omega(b)$ is defined as:

$$\Omega(b) \triangleq 2b^T A y - b^T H b$$

(44)
and

\[ \mathbf{b} = (b_1, b_2, \ldots, b_K)^T \quad (45) \]

\[ \mathbf{H} = \mathbf{A} \mathbf{R} \mathbf{A}^T = A^2 \mathbf{R} \quad (46) \]

### 3.6 (f)

For \( K = 3, \rho_{12} = 0.6, \rho_{13} = 0.4 \), the minimum bit error rate of user 1 is upper bounded by:

\[ P_1(\sigma) \leq \sum_{\epsilon} 2^{1-\omega(\epsilon)} Q \left( \frac{\|S(\epsilon)\|}{\sigma} \right) \quad (47) \]

From equation (36) we can get:

\[
\begin{align*}
P_1(\sigma) &\leq \sum_{\epsilon_2 = 0, -1} \sum_{\epsilon_3 = 0, -1} \cdots \sum_{\epsilon_K = 0, -1} 2^{-\sum_{k=2}^{K} |\epsilon_k|} Q \left( \frac{A \sqrt{1 + \sum_{k=2}^{K} (\epsilon_k^2 + 2 \rho_{1k} \epsilon_k)}}{\sigma} \right) \\
&= Q \left( \frac{A}{\sigma} \right) + \frac{1}{2} Q \left( \frac{2A}{\sqrt{5} \sigma} \right) + \frac{1}{2} Q \left( \frac{\sqrt{6} A}{\sqrt{5} \sigma} \right) + \frac{1}{4} Q \left( \frac{A}{\sigma} \right) \quad (48)
\end{align*}
\]

Here

\[
\begin{align*}
d_{1,\text{min}} &= \sqrt{\left( \frac{A^2}{2} (4 - 2 \rho_{12} - 2 \rho_{13} - |1 - 2 \rho_{12}| - |1 - 2 \rho_{13}|) \right)} \\
&= \sqrt{\frac{4}{5} A^2} \\
&= \frac{2}{\sqrt{5}} A \quad (49)
\end{align*}
\]

This is achieved by \( \epsilon = (1, -1, 0)^T \) (another error vector which can achieve this minimum has the same weight as this one has), which can be seen from the derivation of this minimum. Thus the minimum bit error rate is lower bounded by:

\[
\begin{align*}
P_1(\sigma) &\geq 2^{1-2} Q \left( \frac{d_{1,\text{min}}}{\sigma} \right) \\
&= \frac{1}{2} Q \left( \frac{2A}{\sqrt{5} \sigma} \right) \quad (50)
\end{align*}
\]