1 Basic Results of Gradients (Used in the sequel)

There are 3 basic relations used in the following solutions.
For vectors $a, x \in \mathbb{R}^n$ and symmetric matrix $Q \in \mathbb{R}^{n \times n}$, we have:

\[ \nabla_x \{ a^T x \} = a \]  \hspace{1cm} (1)
\[ \nabla_x \{ x^T a \} = a \]  \hspace{1cm} (2)
\[ \nabla_x \{ x^T Q x \} = 2 Q x \]  \hspace{1cm} (3)

Where the symbol $\nabla_x$ represents the gradient with respect to $x$. And the above results are easy to check, so the proof is omitted here.

Another property of gradient comes from the chain rule of differentiation. We use $[v]_k$ to represent the $k$th component of a vector $v \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$, and let $x_k$ be $x$’s $k$th component. $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is another function. Then we have:

\[ [\nabla_x f(g(x))]_k = \frac{\partial}{\partial x_k} f(g(x)) \]
\[ = \frac{df(v)}{dv} \big|_{v=g(x)} \frac{\partial g(x)}{\partial x_k} \]  \hspace{1cm} (4)

Thus:

\[ \nabla_x f(g(x)) = \frac{df(v)}{dv} \big|_{v=g(x)} \nabla_x g(x) \]  \hspace{1cm} (5)
2 Problem 6.46

Solution: For the simplicity of notation, we use $s$ to represent $s_1$.

2.1 (a)

We now apply Lagrange multipliers to the cost function

$$f(c) = \sum_{j=1}^{n} \lambda^{n-j}(c^Ty[j])^2$$  \hspace{1cm} (6)

Let:

$$g(c, \mu, \nu) = f(c) + \mu(c^Ts - 1) + \nu(c^Tc - 1 - \chi)$$

$$= \sum_{j=1}^{n} \lambda^{n-j}(c^Ty[j])^2 + \mu(c^Ts - 1) + \nu(c^Tc - 1 - \chi) \hspace{1cm} (7)$$

The gradient of this function with respect to $c$ is:

$$\nabla_c g(c, \mu, \nu) = \nabla_c \{ \sum_{j=1}^{n} \lambda^{n-j}c^Ty[j]y^T[j]c \} + \lambda s + 2\nu c$$

$$= 2 \sum_{j=1}^{n} \lambda^{n-j}y[j]y^T[j]c + \lambda s + 2\nu c$$

$$= 2\hat{R}_\lambda[n]c + \lambda s \hspace{1cm} (8)$$

Where symmetric matrix $\hat{R}_\lambda[n]$ is defined as:

$$\hat{R}_\lambda[n] = \sum_{j=1}^{n} \lambda^{n-j}y[j]y^T[j]c + \nu I \hspace{1cm} (9)$$

Let this gradient be zero, we must have:

$$c^* = -\frac{\lambda}{2} \hat{R}_\lambda^{-1}[n]s \hspace{1cm} (10)$$

According to the constraint of $c^{*T}s = 1$, we have:

$$-\frac{\lambda}{2}s^T\hat{R}_\lambda^{-1}[n]s = 1 \iff -\frac{\lambda}{2} = \frac{1}{s^T\hat{R}_\lambda^{-1}[n]s} \hspace{1cm} (11)$$
Plug 11 into 10 we get the final result of $\mathbf{c}$:

$$
\mathbf{c}^*[n] = \alpha[n]\hat{\mathbf{R}}^{-1}_\lambda[n]\mathbf{s}
$$

where $\alpha[n]$ is defined as:

$$
\alpha[n] = \frac{1}{\mathbf{s}^T\hat{\mathbf{R}}^{-1}_\lambda[n]\mathbf{s}}
$$

2.2 (b)

According to 9, we have the expression for the case of $n - 1$:

$$
\hat{\mathbf{R}}_\lambda[n - 1] = \sum_{j=1}^{n-1} \lambda^{n-1-j} \mathbf{y}[j]\mathbf{y}^T[j]\mathbf{c} + \nu\mathbf{I}
$$

Thus:

$$
\hat{\mathbf{R}}_\lambda[n] = \lambda\hat{\mathbf{R}}_\lambda[n - 1] + \mathbf{y}[n]\mathbf{y}^T[n]
$$

According to matrix inversion lemma of (4.99) on page 197, let

$$
\mathbf{A} = \lambda\hat{\mathbf{R}}_\lambda[n - 1]
$$

$$
\mathbf{B} = -\mathbf{I}
$$

$$
\mathbf{C} = \mathbf{y}^T[n]
$$

We have:

$$
\Delta = -1 - \frac{1}{\lambda}\mathbf{y}^T[n]\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n]
$$

$$
\hat{\mathbf{R}}^{-1}_\lambda[n] = \frac{\frac{1}{\lambda}\hat{\mathbf{R}}^{-1}_\lambda[n - 1] - \frac{1}{\lambda}\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n]\mathbf{y}^T[n]\frac{1}{\lambda}\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n]}{1 + \frac{1}{\lambda}\mathbf{y}^T[n]\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n]}
$$

$$
= \frac{1}{\lambda}\hat{\mathbf{R}}^{-1}_\lambda[n - 1] - \frac{1}{\lambda}\beta[n]\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n]\mathbf{y}^T[n]\hat{\mathbf{R}}^{-1}_\lambda[n - 1]
$$

where $\beta[n]$ is defined as

$$
\beta[n] = (\lambda + \mathbf{y}^T[n]\hat{\mathbf{R}}^{-1}_\lambda[n - 1]\mathbf{y}[n])^{-1}
$$
3 Problem 6.51

Solution:

3.1 (a)

We first demonstrate the convexity of the cost function $\Xi(v)$ (let $\lambda \in [0, 1], v_1, v_2 \in \mathbb{R}^K$):

$$
\Xi(\lambda v_1 + (1-\lambda)v_2) = \mathbb{E}\left\{[(\lambda v_1^T + (1-\lambda)v_2^T)r]^2 - 1\right\}
$$

$$
= \mathbb{E}\left\{[\lambda v_1^T r + (1-\lambda)v_2^T r]^2 - 1\right\}
$$

$$
= \lambda^2 \mathbb{E}(v_1^T r)^2 + (1-\lambda)^2 \mathbb{E}(v_2^T r)^2 + 2\lambda(1-\lambda)\mathbb{E}\{v_1^T r \cdot v_2^T r\} - 1
$$

$$
= \lambda \mathbb{E}(v_1^T r)^2 + (1-\lambda)\mathbb{E}(v_2^T r)^2 + 2\lambda(1-\lambda)\mathbb{E}\{v_1^T r \cdot v_2^T r\} - 1
$$

$$
- \lambda(1-\lambda)\mathbb{E}(v_1^T r)^2 - \lambda(1-\lambda)\mathbb{E}(v_2^T r)^2
$$

$$
\leq \lambda \Xi(v_1) + (1-\lambda)\Xi(v_2) - \lambda(1-\lambda)\mathbb{E}\{(v_1 - v_2)^T r\}^2
$$

(23)

since the last term of the last equality is nonnegative.

We now make use of the chain rule 5 to get the gradient of this cost function:

$$
\nabla_v \Xi(v) = \mathbb{E}\nabla_v ((v^T r)^2 - 1)^2
$$

$$
= \mathbb{E}\left\{2((v^T r)^2 - 1)\nabla_v (v^T r r^T v - 1)\right\}
$$

$$
= \mathbb{E}\left\{2((v^T r)^2 - 1) \cdot 2rr^Tv\right\}
$$

$$
= \mathbb{E}\left\{4((v^T r)^2 - 1)rr^Tv\right\}
$$

(24)

According to the principle of stochastic approximation, we can write down the adaptive algorithm for this detector:

$$
v[n] = v[n-1] - \mu((v^T[n-1]r[n])^2 - 1)r[n]r^T[n]v[n-1]
$$

(25)

where $\mu$ is an appropriate step size.
3.2 (b)

In the absence of noise, i.e. $\sigma = 0$, we know $r = SAb$, thus:

$$
r = \sum_{k=1}^{K} A_k b_k s_k \quad (26)
$$

$$
v^T r = \sum_{k=1}^{K} b_k A_k v^T s_k 
= \sum_{k=1}^{K} b_k A_k s_k^T v \quad (27)
$$

Let $y_k \triangleq A_k s_k^T v$, then we know the inner product $v^T r$ cost function can be expressed as:

$$
v^T r = \sum_{k=1}^{K} b_k y_k \quad (28)
$$

$$
Ξ(v) = \mathbb{E}\left\{((\sum_{k=1}^{K} b_k y_k)^2 - 1)^2\right\} 
= \mathbb{E}\left\{\left(\sum_{k=1}^{K} y_k^2 + \sum_{i \neq j} b_i b_j y_i y_j - 1\right)^2\right\} \quad (29)
$$

Notice that since $\{b_j\}_{j=1}^{K}$ are independent and identically distributed with 2 equiprobable values +1, −1, then we have:

$$
\mathbb{E}\left\{2 \sum_{k=1}^{K} y_k^2 \sum_{i \neq j} b_i b_j y_i y_j\right\} = 2 \sum_{k=1}^{K} y_k^2 \sum_{i \neq j} \mathbb{E}\{b_i b_j\} y_i y_j 
= 0 \quad (31)
$$

$$
\mathbb{E}\left\{-2 \sum_{i \neq j} b_i b_j y_i y_j\right\} = -2 \sum_{i \neq j} \mathbb{E}\{b_i b_j\} y_i y_j 
= 0 \quad (32)
$$

$$
\mathbb{E}\left(\sum_{i \neq j} b_i b_j y_i y_j\right)^2 = \mathbb{E}\left(\sum_{i \neq j} b_i b_j y_i y_j\right) \left(\sum_{m \neq n} b_m b_n y_m y_n\right) 
= \sum_{i \neq j} y_i^2 y_j^2 b_i^2 b_j^2 
= \sum_{i \neq j} y_i^2 y_j^2 \quad (33)
$$
Thus we can get our cost function as a function $\Omega(y)$ of $y = (y_1, y_2, \cdots, y_K)^T \in \mathbb{R}^K$:

$$\Omega(y) = \Xi(v(y))$$

$$= \mathbb{E}(\sum_{k=1}^{K} y_k^2)^2 + \mathbb{E}(\sum_{i \neq j} b_i b_j y_i y_j) + 1 + \mathbb{E}\{ -2 \sum_{i \neq j} b_i b_j y_i y_j \} + \mathbb{E}\{ 2 \sum_{k=1}^{K} y_k^2 \sum_{i \neq j} b_i b_j y_i y_j \} - 2 \sum_{k=1}^{K} y_k^2$$

$$= \sum_{i \neq j} y_i^2 y_j^2 + 1 + (\sum_{k=1}^{K} y_k^2)^2 - 2 \sum_{k=1}^{K} y_k^2$$

(35)

Now we turn our attention to the unconstrained minimization problem of the function $\Omega(y)$ which depends on $y = (y_1, y_2, \cdots, y_K)^T \in \mathbb{R}^K$. Obviously this function is nonnegative since it is the expectation of a nonnegative random variable. We claim that 0 is the minimum value of this function (0 is obviously a lower bound of this function). Since this function is obviously convex with respect to $y$, we now let the partial derivatives of this function with respect to each component of $y$ be zero to get the stationary points. And since this minimization problem is unconstrained, we know the global minimum can only be achieved by its stationary points (convexity).

$$\frac{\partial \Omega(y)}{\partial y_k} = 4y_k^3 + 8 \sum_{i \neq k} y_i^2 \cdot y_k - 4y_k$$

$$= 0 \quad \forall \, k = 1, 2, \cdots, K$$

(37)

We now get

$$\begin{align*}
y_k &= 0 \text{ or } y_k^2 + 2 \sum_{i \neq k} y_i^2 = 0 \iff \quad \text{(38)} \\
y_k &= 0 \text{ or } 2 \sum_{i=1}^{K} y_i^2 = 1 + y_k^2 \quad \text{(39)}
\end{align*}$$

Since all components of $y$ are symmetric, we now suppose $m$ out of the $K$ components $y_1, y_2, \cdots, y_m$ satisfy $2 \sum_{i=1}^{K} y_i^2 = 1 + y_k^2$ and other $(K - m)$ components $y_{m+1}, y_{m+2}, \cdots, y_K$ are zeros. This won’t affect out objective to get the minimum of function $\Omega(y)$. Sum the equations 39 over nonzero
components of $y_k$ (i.e. sum from $y_1$ to $y_m$), we can get:

$$
\sum_{k=1}^{m} \left\{ 2 \sum_{i=1}^{K} y_i^2 \right\} = \sum_{k=1}^{m} (1 + y_k^2) \tag{40}
$$

$$
LHS = 2m \sum_{i=1}^{K} y_i^2 \tag{41}
$$

$$
RHS = m + \sum_{i=1}^{m} y_i^2 \tag{42}
$$

$$
= m + \sum_{i=1}^{K} y_i^2 \tag{43}
$$

since $y_{m+1}, y_{m+2}, \ldots, y_K$ are zeros.

From this we obtain:

$$
\sum_{i=1}^{K} y_i^2 = \frac{m}{2m - 1} \tag{44}
$$

Plug 44 into 39 we get:

$$
y_k^2 = \frac{1}{2m - 1} \quad \forall k = 1, 2, \ldots, m \tag{45}
$$

$$
y_k = 0 \quad \forall k = m + 1, m + 2, \ldots, K \tag{46}
$$

Since there are $m$ terms in $\sum_{k=1}^{K} y_k^4$ which are $\frac{1}{(2m-1)^2}$ and others are zeros. There are $m(m - 1)$ terms in $\sum_{i \neq j} y_i^2 y_j^2$ which are $\frac{1}{(2m-1)^2}$ and others are zeros. There are $m$ terms in $\sum_{k=1}^{K} y_k^2$ which are $\frac{1}{2m-1}$ and others are zeros. According to the expression 35, we can calculate the function value for each $m = 1, 2, \ldots, K$:

$$
\Omega(y) = \frac{m}{(2m-1)^2} + 2m(m - 1) \frac{1}{(2m-1)^2} - 2m \frac{1}{2m-1} + 1
$$

$$
= \frac{m - 1}{2m - 1} \tag{47}
$$

$$
= \frac{1}{2} - \frac{1}{4m - 2} \tag{48}
$$

So among the stationary points of $\Omega(y)$, the function value evaluated at these points are monotonically increasing with respect to $m$. The minimum value
of \( m \) is 1 since if \( m = 0 \), the function value can not be calculated as above and \( m = 0 \) indicates \( y_k = 0, \forall k = 1, 2, \cdots, K \) and thus function value is 1 according to 35. But for \( m = 1 \) we have \( \Omega(y) = 0 \), which is the lower bound. Then we know for our cost function to be minimum, it is necessary for only one of the components of \( y_k = A_k s_k^T v \) to be +1 or -1 (Notice here \( m = 1 \), so only one component \( y_k^2 = \frac{1}{2 \times 1 - 1} = 1 \) and others are zeros. Thus the necessary part is proved. The sufficient part is obvious since now we have only one of the components of \( y_k = A_k s_k^T v \) is +1 or -1 and others are zeros. Plug this \( y \) into 35 we get \( \Omega(y) = 0 \), which achieves global minimum.

3.3 (c)

We demonstrate the solution of one of the stationary points and others can be obtained with very similar argument. For our convenience we consider the problem in a 2-dimensional Euclidean space \( \mathbb{R}^2 \) over the real field. This space is isomorphic and isometric to the 2-dimensional Hilbert space of the signal and its complete orthonormal basis is given by \( s_1(t) \) and \( s_2(t) = \rho s_1(t) \sqrt{1 - \rho^2} \).

We now turn this space into \( \mathbb{R}^2 \) to get our discrete-time model. Then the two signature waveforms can be represented by:

\[
\begin{align*}
s_1 &= (1, 0)^T \\
s_2 &= (\rho, \sqrt{1 - \rho^2})^T
\end{align*}
\]  (49)  (50)
All the possible situations for stationary points are (according the result in part (b), \( m = 0, 1, 2 \)):

Solutions for last 2 lines in the table:

\[
A_1 s_1^T v = 1 \quad \text{and} \quad s_2^T v = 0 \quad (51)
\]
\[
A_1 s_1^T v = -1 \quad \text{and} \quad s_2^T v = 0 \quad (52)
\]
\[
A_1 s_1^T v = 0 \quad \text{and} \quad s_2^T v = 1 \quad (53)
\]
\[
A_1 s_1^T v = 0 \quad \text{and} \quad s_2^T v = -1 \quad (54)
\]

Solution for 3rd line in the table:

\[
A_1 s_1^T v = 0 \quad \text{and} \quad s_2^T v = 0 \quad (55)
\]

Solutions for first 2 lines in the table:

\[
A_1 s_1^T v = 1 \quad \text{and} \quad s_2^T v = 1 \quad (56)
\]
\[
A_1 s_1^T v = 1 \quad \text{and} \quad s_2^T v = -1 \quad (57)
\]
\[
A_1 s_1^T v = -1 \quad \text{and} \quad s_2^T v = 1 \quad (58)
\]
\[
A_1 s_1^T v = -1 \quad \text{and} \quad s_2^T v = -0 \quad (59)
\]

(60)

We demonstrate the solution of stationary point when we have:

\[
\begin{align*}
A_1 s_1^T v &= 1 \\
\quad &
\end{align*}
\]
\[
\begin{align*}
s_2^T v &= 0
\end{align*}
\]

These are linear equations of the two components of \( v = (v_1, v_2)^T \in \mathbb{R}^2 \) and can be easily solved and the solution is:

\[
\begin{align*}
v &= \left( \frac{1}{A_1}, -\frac{\rho}{A_1 \sqrt{1-\rho^2}} \right)^T \\
(61)
\end{align*}
\]

Since there’s an isomorphism and isometry between \( \mathbb{R}^2 \) and the signal space, this stationary solution corresponds to signal:

\[
\begin{align*}
v(t) &= v_1 s_1(t) + v_2 \frac{s_2(t) - \rho s_1(t)}{\sqrt{1-\rho^2}} \\
&= \frac{1}{A_1} s_1(t) - \frac{\rho}{A_1 \sqrt{1-\rho^2}} \frac{s_2(t) - \rho s_1(t)}{\sqrt{1-\rho^2}} \\
&= \frac{s_1(t) - \rho s_2(t)}{A_1 (1-\rho^2)} \\
(63)
\end{align*}
\]

This validates the solution in the 3rd line of the table with the positive sign. Other solutions can be obtained in exactly the same way.
To obtain the exact property of the function at these stationary points, we can use the following result:

If $\Omega(y_1, y_2)$ has continuous 2nd-order partial derivatives, let $(y_1^*, y_2^*)^T \in \mathbb{R}^2$ be a stationary point of this function (i.e. 1st-order partial derivatives are both zeros), let

$$a_{11} = \frac{\partial^2 \Omega(y_1, y_2)}{\partial y_1^2} |(y_1, y_2)^T = (y_1^*, y_2^*)^T$$

$$a_{22} = \frac{\partial^2 \Omega(y_1, y_2)}{\partial y_2^2} |(y_1, y_2)^T = (y_1^*, y_2^*)^T$$

$$a_{12} = \frac{\partial^2 \Omega(y_1, y_2)}{\partial y_1 \partial y_2} |(y_1, y_2)^T = (y_1^*, y_2^*)^T$$

Then:

Case 1: If $a_{11}a_{22} - a_{12}^2 > 0$ and $a_{11} > 0$, then $\Omega(y_1^*, y_2^*)$ is a local minimum.

Case 2: If $a_{11}a_{22} - a_{12}^2 > 0$ and $a_{11} < 0$, then $\Omega(y_1^*, y_2^*)$ is a local maximum.

Case 3: If $a_{11}a_{22} - a_{12}^2 > 0$, then $\Omega(y_1^*, y_2^*)$ is neither a minimum nor a maximum.

The stationary points in the last two lines of the table obviously satisfy equations from 51 to 54. These 4 equations are equivalent to the sufficient and necessary condition in part (b), i.e. they achieve global minimum.

For the stationary points in the first two lines of the table, according equations from 56 to 59, we know $y_1^2 = y_2^2 = 1$, and here we have:

$$\Omega(y_1, y_2) = y_1^4 + y_2^4 + 4y_1^2y_2^2 - 2y_1^2 - 2y_2^2 + 1$$

$$\frac{\partial \Omega(y_1, y_2)}{\partial y_1^2} = 12y_1^2 + 8y_2^2$$

$$\frac{\partial \Omega(y_1, y_2)}{\partial y_2^2} = 12y_2^2 + 8y_1^2$$

$$\frac{\partial \Omega(y_1, y_2)}{\partial y_1 \partial y_2} = 16y_1y_2$$

Thus:

$$a_{11} = 20$$

$$a_{22} = 20$$

$$a_{12} = 16$$

We found that $a_{11}a_{22} - a_{12}^2 = 144 > 0$, so the stationary points in the first 2 lines of the table are saddle points.
For the 3rd line in the table which satisfied 55, it is equivalent to $y_1^* = y_2^* = 0$ we prove it is a local maximum. Obviously:

$$\Omega(0,0) = 1$$

$$\lim_{y_1 \to 0, y_2 \to 0} \frac{2(y_1^2 + y_2^2)}{y_1^4 + y_2^4 + 4y_1^2y_2^2} = \infty$$

(72)

According to 67 we know in a neighborhood of $(0,0)^T \in \mathbb{R}^2$, the value 1 is the local maximum.

4 Problem 7.13

Solution: (The answer to this problem should be modified, please see 83) To get the near-far resistance $\eta_{dd}^1$, we just need to minimize the function $\eta_{dd}^1(x) = \min \left\{ 1, (1 - \rho^2)x^2 + \max \left\{ 0, 1 - 2|\rho|x \right\} \right\}$ over $x > 0$. We first discuss the following 2 cases to get the expression for $\eta_{dd}^1(x)$:

Case 1: $1 - 2|\rho|x \geq 0 \iff 0 < x \leq \frac{1}{2|\rho|}$

$$\eta_{dd}^1(x) = \min \left\{ 1, (1 - \rho^2)x^2 + 1 - 4|\rho|x + 4\rho^2x^2 \right\}$$

(73)

There are 2 sub-cases here:

Sub-case 1: $(1 + 3\rho^2)x^2 + 1 - 4|\rho|x \leq 1 \iff 0 < x \leq \frac{4|\rho|}{1 + 3\rho^2}$

Combine the range of $x$ and we get the expression within this range:

$$0 < x \leq \min \left\{ \frac{1}{2|\rho|}, \frac{4|\rho|}{1 + 3\rho^2} \right\}$$

(74)

$$\eta(x) = (1 + 3\rho^2)x^2 + 1 - 4|\rho|x$$

(75)

Sub-case 2: $(1 + 3\rho^2)x^2 + 1 - 4|\rho|x > 1 \iff x > \frac{4|\rho|}{1 + 3\rho^2}$

Combine the range of $x$ and we get the expression within this range:

$$\frac{4|\rho|}{1 + 3\rho^2} < x \leq \frac{1}{2|\rho|}$$

(76)

$$\eta(x) = 1$$

(77)

Notice that in this case in order to validate the range of $x$, we must have $|\rho| \leq \frac{1}{\sqrt{5}}$. 11
Case 2: $1 - 2|\rho| x < 0 \iff x > \frac{1}{2|\rho|}$

$$\eta(x) = \min\{1, (1 - \rho^2) x^2\} \quad (78)$$

There are also 2 sub-cases here:

Sub-case 1: $1 \geq (1 - \rho^2) x^2 \iff 0 < x \leq \frac{1}{\sqrt{1-\rho^2}}$

Combine the range of $x$ and we get the expression within this range:

$$\frac{1}{2|\rho|} < x \leq \frac{1}{\sqrt{1-\rho^2}}$$

$$\eta(x) = (1 - \rho^2) x^2 \quad (80)$$

Notice that in this case in order to validate the range of $x$, we must have $|\rho| > \frac{1}{\sqrt{5}}$.

Sub-case 2: $1 < (1 - \rho^2) x^2 \iff x > \frac{1}{\sqrt{1-\rho^2}}$

Combine the range of $x$ and we get the expression within this range:

$$x > \max\left\{\frac{1}{2|\rho|}, \frac{1}{\sqrt{1-\rho^2}}\right\} \quad (81)$$

$$\eta(x) = 1 \quad (82)$$

In summary, we have:

$$\eta(x) = \begin{cases} 
(1 + 3\rho^2) x^2 + 1 - 4|\rho| x & 0 < x \leq \min\left\{\frac{1}{2|\rho|}, \frac{4|\rho|}{1+3\rho^2}\right\} \\
1 & \frac{4|\rho|}{1+3\rho^2} < x \leq \frac{1}{2|\rho|} (|\rho| \leq \frac{1}{\sqrt{5}}) \text{ or } x > \max\left\{\frac{1}{2|\rho|}, \frac{1}{\sqrt{1-\rho^2}}\right\} \\
(1 - \rho^2) x^2 & \frac{1}{2|\rho|} < x \leq \frac{1}{\sqrt{1-\rho^2}} (|\rho| > \frac{1}{\sqrt{5}})
\end{cases}$$

To minimize $\eta(x)$, we discuss according to the 3 cases above:

1. $0 < x \leq \min\left\{\frac{1}{2|\rho|}, \frac{4|\rho|}{1+3\rho^2}\right\}$

The lower bound is easily achieved at its stationary point $x^* = \frac{2|\rho|}{1+3|\rho|^2}$ and obviously this point is within the range here. The minimum is $\frac{1-\rho^2}{3\rho^2+1}$.

2. $\frac{4|\rho|}{1+3\rho^2} < x \leq \frac{1}{2|\rho|} (|\rho| \leq \frac{1}{\sqrt{5}}) \text{ or } x > \max\left\{\frac{1}{2|\rho|}, \frac{1}{\sqrt{1-\rho^2}}\right\}$

Here the function is a constant 1.

3. $\frac{1}{2|\rho|} < x \leq \frac{1}{\sqrt{1-\rho^2}} (|\rho| > \frac{1}{\sqrt{5}})$

Here $x^2$ has a minimum of $\frac{1}{4\rho^2}$, so under this condition the minimum is $\frac{1-\rho^2}{4\rho^2}$

Compare the 3 minimum values achieved under each situation, we get:

$$\eta_{11}^{dd} = \frac{1 - \rho^2}{3\rho^2 + 1} \quad (83)$$
which is different from the answer in the book. Above is a simulation result for $\rho = 0.5$. 