DIFFERENT EXPRESSIONS OF A MATHEMATICAL OBJECT

John A. Ewell
Department of Mathematics
Northern Illinois University

For given odd primes \( p, q, \ p > q \), the first mathematical object is the number \( N = N(p, q) \) of lattice points on the interior of the rectangle having vertices

\[(0, 0), \ (p/2, 0), \ (p/2, q/2), \ (0, q/2)\]

in the coordinate plane. In the proper setting two different expressions of \( N \) reveal an elegant proof of the celebrated Euler-Legendre-Gauss Law of Quadratic Reciprocity.

The second mathematical object is the infinite product

\[P(x) := \prod_{n=1}^{\infty} (1 + x^{2n-1}),\]

which converges for each complex number \( x \) such that \( |x| < 1 \). Two different expressions of \( P(x) \) reveal a new recurrence for the partition function \( p(\cdot) \). Recall that \( p(\cdot) \) is defined (generated) by the infinite-product expansion

\[\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n,\]

valid for each complex number \( x \) such that \( |x| < 1 \).