Using Imprecise Measures to Study Component and System Reliability

Kimberly F. Sellers
Department of Statistics
Carnegie Mellon University

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Outline

• Binary coherent systems with precise classification
  – Definition
  – Reliability

• Binary state system with imprecise classification
  – Membership functions and probability of fuzzy sets
  – Reliability of degraded components and systems
Binary Coherent Systems

- Range of states, $S=\{0, 1\}$
- State of the $i$th component,
  \[
  X_i = \begin{cases} 
  1, & \text{if component } i \text{ functions} \\
  0, & \text{if component } i \text{ fails}
  \end{cases}
  \]
- Structure function,
  \[
  \phi = \phi(X) = \begin{cases} 
  1, & \text{if the system functions} \\
  0, & \text{if the system fails.}
  \end{cases}
  \]
Binary Coherent Systems (cont.)

- $\phi$ is a binary coherent system if
  1. $\phi$ is nondecreasing
  2. each component is relevant, i.e. $\forall i, \exists (\cdot_i, X) \ni$

$$\phi(1_i, X) = 1 \text{ and } \phi(0_i, X) = 0.$$
Binary Coherent Systems (cont.)

Basic examples of binary coherent systems:

- **Series:**

\[
\phi(X) = \prod_{i=1}^{n} X_i
\]

- **Parallel (not stand-by):**

\[
\phi(X) = \prod_{i=1}^{n} X_i = 1 - \prod_{i=1}^{n} (1 - X_i)
\]

- **k-out-of-n:**

\[
\phi(X) = \begin{cases} 
1, & \text{if } \sum_{i=1}^{n} X_i \geq k \\
0, & \text{if } \sum_{i=1}^{n} X_i < k
\end{cases}
\]
Reliability of Binary Coherent Systems

- \( X_i \mid p_i \sim \text{Bernoulli}(p_i) \)
  - \( p_i = P(X_i = 1 \mid p_i) \) - reliability of component \( i \)
  - \( h(p) = P(\phi(X) = 1 \mid p) \) - reliability of \( \phi \)
  - System reliability is determined by the reliability of its components

- Assumptions:
  - Given \( p_i \), \( X_i \perp X_j \ \forall i \neq j \).
  - Given \( p_i \), \( X_i \perp p_j \ \forall j \neq i \).
Reliability of Binary Coherent Systems (cont.)

• Series system:

\[ h_S(p) = P(\phi(X) = 1 \mid p) = \prod_{i=1}^{n} p_i \]

• Parallel (not stand-by) system:

\[ h_P(p) = \prod_{i=1}^{n} p_i = 1 - \prod_{i=1}^{n} (1 - p_i) \]

• k-out-of-n system:

\[ h_K(\tilde{p}) = \sum_{j=k}^{n} \binom{n}{j} \tilde{p}^j (1 - \tilde{p})^{n-j} \]

if all components are identical with reliability \( p_i \equiv \tilde{p} \).
Binary State Systems with Imprecise Classification

- Membership functions and probability of fuzzy sets
- Reliability of degraded components and systems
Membership Function, $\mu_A(x)$

- For each $x$, $0 \leq \mu_A(x) \leq 1$ describes a belief of containment (membership) in a category $A$.

- If $\mu_A(x) = 0$ or $1$, then $A$ is a precise (i.e., crisp) set. Otherwise, $A$ is a fuzzy set.

**Ex. 1:** $A_1 = \{x \in (1, 2, \ldots, 10) \mid x \geq 7\}$

<table>
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<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{A_1}(x)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Ex. 2:** $A_2 = \{x \in (1, 2, \ldots, 10) \mid x$ is large$\}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0.5</td>
<td>0.9</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Membership Function, $\mu_A(x)$

- Operations:
  1. $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
  2. $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
  3. $\mu_{A^c}(x) = 1 - \mu_A(x)$

- Definitions parallel Lukasiewicz’s (1930) definitions for the conjunction, disjunction and complement of many-valued propositions

- Bellman and Giertz (1973) showed that Operations 1 and 2 are unique with respect to constructing an algebra of sets; Operation 3 is not unique but “appears reasonable”.
Probability of Fuzzy Sets

Due to Bement et. al. (2000)

• Based on two premises: $\mu_{\tilde{A}}(x)$ is interpreted as a likelihood function, and fuzzy sets arise from boundary uncertainty

• $\pi_D(x \in \tilde{A})$ is $D$’s probability that $x \in \tilde{A}$ if it is known that $X = x$

• $\mu_{\tilde{A}}(x)$ is additional information (expert testimony) provided by expert, $Z$

\[
P_D(X \in \tilde{A}; \mu_{\tilde{A}}(x)) \propto \int_x \mu_{\tilde{A}}(x)\pi_D(x \in \tilde{A})dP_D(x) \tag{1}
\]
Probability of Fuzzy Sets (cont.)

- $D$ needs to specify two probability measures, $P_D$ and $\pi_D$
  - $(\Omega, \mathcal{F}, P_D)$, where $\Omega$ is the set of all possible outcomes of the random phenomenon under study
  - $(\Omega, \mathcal{F}, \pi_D)$, where $\Omega$ consists of two outcomes $x \in \tilde{A}$ or $x \notin \tilde{A}$
  - $\pi_D(x \in \tilde{A}) = 1 - \pi_D(x \notin \tilde{A})$

- To evaluate the constant of proportionality, $D$ needs to evaluate
  \[
P_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x)) \propto \int_x \mathcal{L}_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x))\pi_D(x \notin \tilde{A})dP_D(x) \quad (2)\]

- $\mathcal{L}_D(X \notin \tilde{A}; \mu_{\tilde{A}}(x))$ not necessarily $1 - \mu_{\tilde{A}}(x)$
Components in Vague Binary States

- Let $X$ = the state of a component at time $\tau > 0$
- $X$ takes values in $S = \{x; \ 0 \leq x \leq 1\}$
- Consider $S_1 \subset S$, where $S_1 = \{x; \ x$ is a “desirable” state$\}$ and $
\mu_1(x)$ membership function
- If $S_2 \subset S$ was defined as

$$S_2 = \{x; \ x$ is an “undesirable” state$\},$$

then $S_1^C$ need not be $S_2$ unless $\mu_2(x) = 1 - \mu_1(x)$, where $\mu_2(x)$
is the membership function of $S_2$
Components in Vague Binary States (cont.)

- $\mu_2(x)$ need not be symmetrical to $\mu_1(x)$

Symmetric case  
Asymmetric case

- In general, $S_1^C$ need not be $S_2$ and vice versa, unless $S_1$ and $S_2$ are precise sets.
Components in Vague Binary States (cont.)

- For $S_1=\{x; \ x \text{ is a “desirable” state}\}$ and $\mu_1(x)$ specified, we define component reliability as

$$P_D(X \in S_1; \mu_1(x)) \propto \int_x \mu_1(x)\pi_D(x \in S_1)dP_D(x). \quad (3)$$

- With $S_2=\{x; \ x \text{ is an “undesirable” state}\}$, and $\mu_2(x)$ specified, we define component unreliability as

$$P_D(X \in S_2; \mu_2(x)) \propto \int_x \mu_2(x)\pi_D(x \in S_2)dP_D(x). \quad (4)$$
Components in Vague Binary States (cont.)

- We could define component unreliability as
  \[ P_D(X \in S_1^C; \mu_1(x)) \], where \( S_1^C \) has membership function, 
  \( 1 - \mu_1(x) \). In this case, the component unreliability is

  \[ P_D(X \in S_1^C; \mu_1(x)) \propto \int_x (1 - \mu_1(x))[1 - \pi_D(x \in S_1)]dP_D(x). \] (5)

- When a component’s state is vague and binary, its unreliability is not the complement of its reliability!
Vague Binary Systems

• Claim: the structure functions of binary state coherent systems with precise classification are membership functions of precise sets

• Let $X_i =$ state of component $i$, taking values in $S = \{x; \ 0 \leq x \leq 1\}$

• $S_1^{[i]} \subset S$, where $S_1^{[i]} = \{x; \ x_i \text{ is a “desirable” state}\}$ with associated membership function, $\mu_1^{[i]}(x), \ i = 1, \ldots, n$

• Consider $S_1^{[i]} \subset S$ precise $\forall i$. e.g., $\forall i, \ \exists x_i \ni$

$$\mu_1^{[i]}(x) = \begin{cases} 1 & x \geq x_i^* \\ 0 & x < x_i^* \end{cases}$$
Vague Binary Systems

- Let $\mathbf{X} = (X_1, \ldots, X_n)$ and suppose that the $n$ components are connected in series. Thus $\phi_S(\mathbf{X}) = 1$ if and only if $x \geq x_i^*$ for all $i$, $i = 1, \ldots, n$, implying

$$
\phi_S(\mathbf{X}) = \prod_{i=1}^{n} \mu_1^{[i]}(X_i) = \min_i [\mu_1^{[i]}(X_i)], \tag{6}
$$

- Similarly, if $n$ components were connected in parallel redundancy, then

$$
\phi_P(\mathbf{X}) = \prod_{i=1}^{n} \mu_1^{[i]}(X_i) = \max_i [\mu_1^{[i]}(X_i)] \tag{7}
$$

- For a $k$-out-of-$n$ system, $\phi_K(\mathbf{X}) = \mu_1^{[(n-k+1)]}(\mathbf{X})$ where $\mu_1^{[(n-k+1)]}(\mathbf{X})$ represents the $(n - k + 1)$st membership function when $\mu_1^{[i]}(X_i)$'s are ordered
Vague Binary Systems

Motivated by the above, we define system structure functions as

$$
\phi_S(X) = \min_i [\mu_1^{[i]}(X_i)] = \mu_1^{[1]}(X),
$$

$$
\phi_P(X) = \max_i [\mu_1^{[i]}(X_i)] = \mu_1^{[n]}(X), \text{ and}
$$

$$
\phi_K(X) = \mu_1^{[(n-k+1)]}(X)
$$
Vague Binary Systems

There are two different strategies for defining the reliability of a vague coherent system.

1. Assume that a system is reliable if the necessary components are in a desirable state

   • The reliability of a series system is \( \prod_{i=1}^{n} [P_{D}(X_i \in S_{1}^{[i]}; \mu_{1}^{[i]}(x_i))] \) assuming independent \( X_i \)'s, where

     \[
P_{D}(X_i \in S_{1}^{[i]}; \mu_{1}^{[i]}(x_i)) \propto \int_{x_i} \mu_{1}^{[i]}(x_i) \pi_{D}(x_i \in S_{1}^{[i]}) dP_{D}^{[i]}(x_i), \quad (8)
     \]

   • Assuming independent \( X_i \)'s, the reliability of a parallel redundant system is

     \[
P_{D}\left( \bigcup_{i=1}^{n} \{X_i \in S_{1}^{[i]}\}; \mu_{1}^{[i]}(x_i), \; i = 1, \ldots, n \right)
     \]

   • The case of \( k \)-out-of-\( n \) systems follows along similar lines
Vague Binary Systems

2. \( S^*_\phi(x) = \{x; \phi(X) = x \text{ is a “desirable” state of the system}\} \subset S \)

with associated membership function \( \mu^*_\phi(x) \).

- The reliability of a series system is

\[
P_D(\phi_S(X) \in S^*_\phi(x); \mu^*_\phi(x)) \propto \int_{x} \mu^*_\phi(x) \pi_D(x \in S^*_\phi(x)) \, dp_D(x),
\]

where \( dp_D(x) \) is obtained from

\[
P_D(\phi_S(X) \geq x) = \prod_{i=1}^{n} P_D(X_i \geq \mu_1^{[i]-1}(x)),
\]

assuming independent \( X_i \)'s; \( \mu_1^{[i]-1}(x) \) denotes the inverse of \( \mu_1^{[i]}(x) \).
Vague Binary Systems (cont.)

- Similarly for parallel systems,

\[ P_D(\phi_P(\mathbf{X}) \in S_{\phi_P}^*(\mathbf{x}); \mu_{\phi_P}^*(\mathbf{x})(x)) \propto \int_x \mu_{\phi_P}^*(\mathbf{x})(x) \pi_D(x \in S_{\phi_P}^*(\mathbf{x})) dP_D(x), \]

where \( dP_D(x) \) is obtained via \( \prod_{i=1}^n P_D(X_i \leq \mu_1^{[i]-1}(x)) \) assuming independent \( X_i \)'s.

- The case of \((n - k + 1)\)-out-of-\(n\) follows by considering the distribution of the \(k\)th order membership function, \( \mu_1^{[(k)]}(\mathbf{X}) \).
Ongoing Research

- Reliability of \((m + 1)\)-level degraded systems
- Other applications
  - Quality of life
  - Disclosure limitation
References


References (cont.)


