THE PRELIMS FORMULA SHIIIIIIIT

XIN XIANG

Dear Reader: this is a condensed version of review notes which I compiled for my own prelims study. So please forgive me for the lack of explanation. Contact me (xxiang@) if you found anything terribly wrong! Good luck!

0. Handy Math Facts

Triple Product: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \)

Second Derivative: \( \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \)

Exponential (BCH): \( e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \)

Orbital acceleration in polar: \( \mathbf{a} = (\dot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \)

Legendre polynomials: \( P_l(x) \) = 1, \( P_2(x) = x, P_3(x) = (3x^2 - 1)/2; \) orthogonality: \( \int_{-1}^1 P_m(x)P_n(x)dx = \frac{2}{2m+1}\delta_{mn} \)

Eccentricity: \( e > 1 \) hyperbola; \( e = 1 \) parabola; \( 0 < e < 1 \) ellipse; \( e = 0 \) circle

Stirling’s Approximation: \( \ln N! = N \ln N - N \)

Gaussian Integral: \( \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\pi/a} \) \( \int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \sqrt{\pi/2a^{3/2}} \) \( \int_{-\infty}^{\infty} x^3 e^{-ax^2} = 1/2a^2 \)

Finite geometric sum: \( \sum_{k=0}^{n-1} a^k = \frac{a(1-r^n)}{1-r} \)

Memorize the metric \( \left( \begin{array}{ccc}
\gamma & -\beta & 0 \\
-\beta & \gamma & 0 \\
0 & 0 & 1
\end{array} \right) \)

Lorentz boost in \( +\hat{x} \):

1. CLASSICAL MECHANICS

1.1. Small Oscillation. J08M1, J06M3, M05M1, M05M3, M04M2, J04M1, Cahn1.17, Cahn1.26

\[ L = \frac{1}{2} T_{ij} q_i' q_j' - \frac{1}{2} V_{ij} q_i q_j \] where \( q' = q - \phi_0 \). To find eigenfrequencies, set \( \det(\tilde{\mathbf{V}} - \omega^2 \mathbf{T}) = 0 \); to find eigenmodes, solve \( \tilde{\mathbf{V}} \cdot \mathbf{a} = \omega^2 \mathbf{T} \cdot \mathbf{a} \).

1.2. Damped oscillator. blah

\[ m \dddot{x}(t) + b \dot{x}(t) + kx(t) = 0 \] is solved by guessing \( x(t) \sim e^{\lambda t} \).

1.3. Central potential. J06M2, J04M2, J03M3, M97M2, Cahn1.14, Cahn1.15, Cahn1.16

Solving orbit: substitute \( u = 1/r \) and use \( \frac{d}{dt} = \frac{1}{mr^2} \frac{d}{dr} \). EOM is then \( \ddot{u} + u = -\frac{m}{r^2} \frac{dV}{du} \). Only Kepler and Harmonic potential has closed orbit.

1.4. Non-slip rolling. J08M2, J07M1, M06M1, M03M2, M99M1, J99M1, Cahn1.41

Constrain: \( \dot{\mathbf{r}} + \alpha \mathbf{n} \times \mathbf{w} = 0 \)

Trick: equation of motion toque at point of contact.
1.5. Hydrodynamics. J08M3, J07M3, M03M3, J01M2, M98M3, Cahn1.12
Bernoulli (incompressible) equation: \( \frac{1}{2} \rho u^2 + \rho gh + p = \text{const} \)
Continuity equation: \( \frac{\partial \rho}{\partial t} + \nabla (\rho v) = 0 \)
Navier stokes equation: \( \rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -\nabla p + \rho \nabla^2 v = \mathbf{f} \), where \( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \text{material derivative} \)
Surface tension: \( F = \lambda l \) where \( l \) is length where force is acting

1.6. Waves/string/Spring. M07M3, M06M3, J05M3, M02M2, J02M3, M01M1, Cahn1.46
Young’s modulus: \( E \equiv \frac{F/A_o}{\Delta L/L_o} \)

1.7. Rigid body motion. J06M1, M05M3, M03M1, M02M3, J01M2, J01M3, M00M3, Cahn1.10, Cahn1.33
Momentum of Inertia: \( I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_i \alpha r_i \beta) \)  
Ex: \( I = \frac{1}{2} MR^2 \) (Disk & Cylinder); \( I = \frac{2}{3} MR^2 \) (spherical shell); \( I = \frac{2}{5} MR^2 \) (solid sphere); \( I = \frac{1}{2} M [w^2 + h^2] \) (rectangular)  
Parallel axis theorem: \( I = I_m + ma^2 \).
For any quantities: \( \left( \frac{dx}{dt} \right)_s = \left( \frac{dx}{dt} \right)_b + \omega \times \mathbf{x} \) where \( s \) means space frame and \( b \) means body frame.
Euler’s equations: \( \frac{dV}{dt} = \omega_2 \omega_3 (I_2 - I_3) \) and permute \( 1 \to 2 \to 3 \) you get others
EOM in rotating frame: \( \mathbf{a}_b = \mathbf{a}_f - \mathbf{a}_w \cdot \omega \times (\omega \times \mathbf{r}) \) where the last two terms are coriolis and centrifugal effect.
Kinematic: \( T = \frac{1}{2} \omega \cdot \mathbf{L} = \frac{1}{2} \mathbf{L} \cdot \mathbf{I}^{-1} \mathbf{L} \); it is easier to calculated it in body frame.

1.8. Miscellaneous. blah
Hydrostatic equilibrium: \( \frac{dP}{dr} = -\rho(r)g(r) \)

2. ELECTRICITY MAGNETISM

2.1. Method of images. M07E1, M03E2, J03E2, M00E2
Inversion relation \( (V(R) = 0) \): \( q' = -\frac{R}{a} q \) and \( b = \frac{R^2}{a} \). If the spherical conductor has a net charge of \( Q \), add another imaginary charge \( Q - q' \) at the center. If the spherical conductor has a constant potential \( V_0 \), add a charge \( (4\pi \epsilon_0) V_0 R \) at the center. In the presence of two spherical conductors, inversion \( (a - b) b = R^2 \) gives \( b \) in closed form, and expansion around \( R/a \sim 0 \) gives \( n \)th reflections. If it’s a half dielectric space, image charge locates at -\( x \) with charge \( q' = \frac{1}{2} \epsilon_\infty \frac{q}{1 + \epsilon_\infty} \) which can be showed from boundary condition; for potential in dielectric, \( V = \frac{1}{4\pi \epsilon_0} \frac{q''}{r} \) where \( q'' = \frac{2q}{1 + \epsilon_\infty} \) locates at \( q \).

2.2. Capacitors. J05E1, J04E1, M01E1, J00E1, J97E1, M96E2, M95E1, J95E1
\( C = Q/V \) and yeah, just do it!

2.3. Electrostatic. blah
Solution to spherical Laplacian: \( V(r, \theta) = \sum_{n=0}^{\infty} \left( A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta) \)
Potential: \( V = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{r} d\tau \)  
\( A = \frac{\mu_0}{4\pi} \int \frac{\hat{P}(r') \cdot \hat{r}}{r^2} d\tau \)  
\( \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \)
Dipole: \( p = \int \rho(r) r d\tau \)  
\( V_{\text{dip}} = \frac{1}{4\pi \epsilon_0} \frac{p}{r} \)  
\( \mathbf{E}_{\text{dip}} = \frac{1}{4\pi \epsilon_0} \frac{1}{r^2} [3(p \cdot \hat{r}) \hat{r} - p] \)

2.4. Magneto-static. blah
Biot-Savart Law: \( \mathbf{B}(r) = \frac{\mu_0}{2\pi} \int \frac{dx \hat{n}}{r} \)  
\( \mathbf{B} = \nabla \times \mathbf{A} \)
Force on dipole: \( \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \)  
Mag dipole: \( \mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{m \times \hat{r}}{r^2} \)  
\( \mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{1}{r^2} [3(\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m}] \)
Volume current: \( \mathbf{J}_b = \nabla \times \mathbf{M} \)  
Surface current: \( \mathbf{K}_b = \mathbf{M} \times \hat{n} \)
Tricks: magneto static scalar potential \( \mathbf{B} = -\nabla U \) can be used as a theoretical tool if it’s a simply connected current free region.

2.5. Electric Magnetic field in matter. M00E2, J97E1
Bound density charge: \( \sigma_b = \mathbf{P} \cdot \hat{n} \)  
\( \rho_h = -\nabla \cdot \mathbf{P} \)
Linear dielectric: \( \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \) and \( \epsilon_r = (1 + \chi_e) = \epsilon/\epsilon_0 \), and \( \nabla \cdot \mathbf{D} = \rho_f \)
Boundary conditions: \( D_1^+ - D_2^- = \sigma_f \)  
\( E_1^+ = E_2^- \)  
\( B_1^+ = B_2^- \)  
\( \mathbf{H}_1^+ - \mathbf{H}_2^- = \mathbf{K}_f \times \hat{n} \)
Auxiliary field: \( \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} = \frac{1}{\mu} \mathbf{B} \)
2.6. **Electrodynamics.** blah

Self-Inductance: \( \Phi = LI \); emf = \(-L \frac{di}{dt}\); \( E = \frac{1}{2} LI^2 \)  
Mutual inductance: \( \Phi_2 = M_{21} I_1; M_{12} = M_{21} \)

Maxwell in vacuum  
\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \rho / \varepsilon_0 \\
\nabla \times \mathbf{E} &= \frac{-\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Maxwell in matter  
\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_f \\
\nabla \times \mathbf{E} &= \frac{-\partial \mathbf{H}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

2.7. **Energy and momentum.** blah

Energy density: \( u_{em} = \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \)  
Poynting’s theorem: \( \frac{\partial}{\partial t} (u_{mech} + u_{em}) = -\nabla \cdot \mathbf{S} \)

Maxwell stress tensor: \( T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \)

Momentum density: \( p_{em} = \mu_0 \varepsilon_0 \mathbf{S} \)  
\( \frac{\partial}{\partial t} (p_{mech} + p_{em}) = \nabla \cdot \mathbf{T} \)

2.8. **Gauge.** blah

Coulomb gauge: \( \nabla \cdot \mathbf{A} = 0 \)  
Lorentz gauge: \( \nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial \mathbf{V}}{\partial t} \)

2.9. **Radiation.** M06E3, J06E3, M05E3, J05E2, M04E2, M04E1, J03E3, J03E2, M00E2, J00E3, M99E1

Retarded potential: same as static case except the \( t_r = t - n/c \) dependence in \( \rho \) and \( \mathbf{J} \)

Electric Dipole: \( \mathbf{E} = \frac{\mathbf{p}}{\varepsilon_0 c} \left( \mathbf{r} \times \left( \mathbf{r} \times \mathbf{p}(t_r) \right) \right); \mathbf{B} = \mathbf{r} \times \mathbf{E} / c \)

If \( \mathbf{p} = p_0 \cos \omega t \), then \( \mathbf{E} = -\frac{\mu_0 p_0}{4\pi} \left( \sin \theta \right) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \hat{\mathbf{r}} \)  
\( \langle \mathbf{S} \rangle \sim p_0^2 \omega^4 \sin^2 \theta \frac{\hat{\mathbf{r}}}{r^2} \) (a donut); \( \langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \)

Magnetic dipole: substitute \( p_0 \to m_0 / c \); \( \mathbf{E} \) is on \( \hat{\theta} \) direction and \( \mathbf{B} \) is on \(-\hat{\theta}\)

Point charge: \( \mathbf{E} = \frac{\mu_0 q}{4\pi} \left( \frac{1}{n} \mathbf{n} \times \left( \mathbf{n} \times \hat{\mathbf{a}} \right) \right); \mathbf{S} = \frac{1}{\mu_0 c} E^2 \mathbf{n} \)

Larmor formula: \( P = \frac{\mu_0 q^2 a^2}{6\pi c} \), where \( a \) is acceleration in the rest frame of charge. If \( \mathbf{a} \cdot \mathbf{u} = 0 \), \( a_{lab} = a \gamma^2 \)

Abraham-Lorentz formula (radiation reaction): \( \mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c a} \hat{\mathbf{a}} \)

2.10. **Electromagnetic waves.** M07E3, J07E3, J05E2, J03E1, J02E1, J01E3, M00E1, J99E1, J99E2

Oblique Incidence: \( E_R = \left( \frac{\alpha - \beta}{\alpha + \beta} \right) E_I \); \( E_T = \left( \frac{2}{\alpha + \beta} \right) E_I \) where \( \alpha = \frac{\cos \theta_f}{\cos \theta_i} \). Derive it from B.C.

Wave guide: generic form \( \mathbf{E} = E_0(x,y) e^{i(kz-\omega t)} \); \( \mathbf{B} = B_0(x,y) e^{i(kz-\omega t)} \). Inner wall B.C. \( \mathbf{E} \parallel = 0 \); \( B \perp = 0 \); TE: \( E_z = 0 \); TM: \( B_z = 0 \); TEM: \( E_x = B_x = 0 \). TEM cannot occur in hollow waveguide. TEM in coaxial cable has \( \nabla \cdot \mathbf{E} = 0 \); \( \nabla \cdot \mathbf{B} = 0 \) and therefore is \( \mathbf{E} = \frac{A}{s} \hat{s}; \mathbf{B} = \frac{A}{cs} \hat{\phi} \)

2.11. **Relativistic EM.** J09E1

Faraday Tensor \((+,-,-,-); F^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)  
\[
\begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_2 & -B_2 & B_1 & 0
\end{pmatrix}
\]
where \( A_\mu = (\phi, \mathbf{A}) \).

3. **QUANTUM MECHANICS**

3.1. **Commutation identities.** blah

\([p_i, F(x)] = -i\hbar \frac{\partial F(x)}{\partial x_i} \); \([x_i, F(p)] = i\hbar \frac{\partial F(p)}{\partial p_i} \); \([p, x^n] = -i\hbar n x^{n-1} \)

Heisenberg EOM: \( \frac{dQ}{dt} = \frac{i}{\hbar} [H, Q] + \frac{\partial Q}{\partial t} \)  
Uncertainty principle: \( \sigma_D^2 \sigma_B^2 \geq \left( \frac{1}{2\hbar} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \)

3.2. **Some Hamiltonians.** blah

\( H = \frac{1}{2m} \left( \mathbf{p} - q \mathbf{A}/c \right)^2 + q \phi \)

3.3. **Harmonic Oscillator.** blah

Memorized by heart: \( H_n = \hbar \omega \left( \frac{n}{2} + 1 \right); [a, a^+] = 1; [a, |n\rangle] = \sqrt{n} |n-1\rangle; [a^+, |n\rangle] = \sqrt{n+1} |n+1\rangle; [a^+ a, |n\rangle] = n |n\rangle \)

\( \hat{H} = \hbar \omega \left( \frac{1}{2} + a^+ a \right) \); \( a^+ = \sqrt{\frac{m \omega}{2\hbar}} (\hat{x} + \frac{i}{m \omega} \hat{p}); \hat{x} = \sqrt{\frac{\hbar}{m \omega}} (a + a^+); \hat{p} = i \sqrt{\frac{m \omega}{2\hbar}} (a^+ - a) \)

The wavefunctions: \( \psi_0 = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m \omega}{2\hbar} x^2}; |n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle \)
3.4. Boundary Conditions. blah
\[ \psi(x) \] is continuous at boundary, \( \psi'(x) \) is continuous at finite boundary. For \( V = \alpha \delta(x) \), \( \Delta \left( \frac{d\psi}{dx} \right) = \frac{2m\alpha}{\hbar^2} \psi(0) \)

3.5. 3D Schroedinger, Hydrogen and Others. blah
Radial equation: \( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \left( \frac{\gamma}{r} - \frac{\lambda^2}{r^2} \right) \psi = \frac{-\mu}{\hbar^2} \psi \)
Ground State: \( \frac{1}{\pi a_0^2} e^{-r/a_0} \), where \( a_0 = \frac{\hbar}{\alpha m \omega_c} \approx 0.5 \AA \)
Useful: \( \langle \frac{1}{r} \rangle = \frac{1}{n^2 a_0^2} \), \( \langle \frac{1}{r^2} \rangle = \frac{1}{(n(1/2m)^2 a_0^2) \rangle} \)
K-shell screening: \( E_n = (-13.6eV) \frac{n^2 - 1}{n^2} \) Selection rules: \( \Delta m = \pm 1 \) or 0 and \( \Delta l = \pm 1 \)

3.6. Angular momentum. blah
Angular momentum: \( [J_x, J_y] = i\hbar J_z; [J_z, J_\pm] = 0; J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle; J_z |jm\rangle = h m |jm\rangle; J = J_x \pm \imath J_y \)
Addition of 2: \( J_\pm = J_1 \pm J_2 \pm \)
Pauli Spin: \( S^2 = \frac{3\hbar^2}{4} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \sigma_x = \left( \begin{array}{cc} 0 & -\imath \\ -\imath & 0 \end{array} \right) ; \sigma_y = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) ; \sigma_z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \)
Addition of two spin 1/2: \( |11\rangle = |++\rangle; |10\rangle = \frac{1}{\sqrt{2}} (|+-\rangle + |-+\rangle); |1-\rangle = |--\rangle; |00\rangle = \frac{1}{\sqrt{2}} (|++\rangle - |--\rangle) \)
Ladder: \( J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |jm \pm 1\rangle \) where the normalization is obtained from bracketing \( J_- J_+ = J^2 - \frac{1}{2} J_z^2 - J_z \) with \( \langle m_j | \) and \( |m_j \rangle \)

3.7. Two identical particle. blah
Exchange operator: \( P \psi(r_1, r_2) = \psi(r_2, r_1) \) has eigenvalue \( \pm 1 \) with +1 corresponds to boson and -1 to fermion. \( \psi_\pm(r_1, r_2) = A \psi_a(r_1) \psi_b(r_2) \pm \psi_b(r_1) \psi_a(r_2) \)
Hund’s rules (apply them in order): 1. The state with the highest total spin \( S \) minimize the energy. 2. The state with the largest total orbital angular momentum \( L \) minimize the energy; 3. if a subshell is no more than half filled, the lowest energy level has \( J = |L - S| \), if more than half filled, \( J = L + S \) has the lowest energy. \( 2S+1L_J \)

3.8. The variational principle. blah
Minimize \( \langle H \rangle \) respect to parameters to find optimal ground state and energy.

3.9. WKB approximation. blah
number of wavelength = \( \int_a^b \frac{dx}{x(\pi x)} = \frac{n}{2} + \frac{1}{4} \) where \( n \) is half wavelength number
Tunneling: \( T = e^{-2\gamma} \) where \( \gamma = \frac{1}{\hbar} \int_a^b |\psi(x)| dx \)

3.10. Time-independent Perturbation. blah
Non-degenerate: \( E_i^1 = \langle \psi^0_n | H' | \psi^0_n \rangle; \psi^1_n = \sum_{m \neq n} \frac{\langle \psi^0_m | H' | \psi^0_n \rangle}{E_m^0 - E_n^0} \psi^0_m; E_i^2 = \sum_{m \neq n} |\langle \psi^0_m | H' | \psi^0_n \rangle|^2 \)
Degenerate: pick a basis of eigenstates, and diagonalize \( H \)! Above formulas only work for those do not live in degenerate subspace.

3.11. Time-dependent perturbation. blah
Transition probability: \( P_{i\rightarrow f} = \frac{|\psi_i | |E_f - E_i | \hbar}{2} \left( \langle f | V(t) | i \rangle \right)^2 \) Transition rate: \( R_{i \rightarrow f} = \frac{2\pi}{\hbar} | \langle f | V | i \rangle |^2 \rho \) where \( \rho \) is the density of final state

3.12. Scattering. blah
Memorized by heart: \( \psi \approx A \left( e^{ikx} + f(\theta, \phi) e^{ikr} \right) \); \( \frac{dr}{d\Omega} = |f(\theta, \phi)|^2 \)
Born Approximation: \( f(\theta, \phi) \approx -m \frac{e^{i\kappa r_0}}{2\pi \hbar^2} \int e^{i\kappa r_0} V(r_0) d^3 r_0 \) where \( \kappa = k - k' \); if potential is spherical symmetry, \( f(\theta, \phi) \approx -2m \frac{e^{i\kappa r_0}}{2\pi \hbar^2} \int_0^{2\pi} r V(r) \sin(\kappa r) dr \) is the momentum transfer. Born approximation is only valid if incident energy is larger than the scattering potential, so the deflection is small.
4. Statistical Mechanics

4.1. Thermodynamical identities. blah
Many identities can be derived from the following table. For example, \( p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \). Maxwell relations are obtained by taking derivative twice

<table>
<thead>
<tr>
<th>Potential</th>
<th>Fundamental Differential Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal energy ( U(S,V,n_i) )</td>
<td>( dU = TdS - pdV + \sum\mu_i dN_i )</td>
</tr>
<tr>
<td>Enthalpy ( H(S,p,n_i) \equiv U + pV )</td>
<td>( dH = TdS + Vdp + \sum\mu_i dN_i )</td>
</tr>
<tr>
<td>Helmholtz Free ( F(T,V,n_i) \equiv U - TS )</td>
<td>( dF = -SdT - pdV + \sum\mu_i dN_i )</td>
</tr>
<tr>
<td>Gibbs Free ( G(T,p,n_i) \equiv H - TS )</td>
<td>( dG = -SdT + Vdp + \sum\mu_i dN_i )</td>
</tr>
</tbody>
</table>

Other definitions: \( S \equiv k_B \ln \Omega \) \( C_p = \left( \frac{\partial U}{\partial T} \right)_p + p \left( \frac{\partial V}{\partial T} \right)_p \) \( \chi = \left( \frac{\partial M}{\partial H} \right)_B = 0 \)

Useful relations: \( F = -k_B T \ln Z; S(T) = \int_0^T C_V(T')/T' dT' \)

4.2. Partition function. blah
If Hamiltonian \( H \) is separable, \( H = H_1 + H_2 \), then \( Z = Z_1 Z_2 \). For \( N \) identical particles, \( Z = \frac{1}{N!} Z_1^N \). A quick way to get energy from partition function: \( E = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{\partial}{\partial \beta} \ln Z \)

4.3. Ideal gas. blah
Partition function: \( Z_{\text{tran}} = \frac{1}{N!} \left( \frac{V}{N} \right)^N \) where \( \lambda_d = h/\sqrt{2\pi mk_BT} \); diatomic: \( Z_{\text{rot}} = \frac{2Bk_BT}{k^2} \)

\( v_{rms} = \sqrt{\frac{3k_BT}{m}}, \bar{v} = \sqrt{\frac{8k_BT}{\pi m}} \)

Entropy (Micro, surface of N sphere): \( S = Nk_B \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi mU}{3Nk_BT} \right)^{3/2} \right) + \frac{5}{2} \right] \)

Adiabatic process: \( PV^\gamma = \text{const} \) where \( \gamma = C_p/C_V = (f + 2)/f \)

4.4. Paramagnetism. blah
Partition functions: \( Z = 2 \cosh(\beta \mu B) \)

4.5. Boson and Fermion. blah
Bose-Einstein distribution: \( \bar{n}_{BE} = \frac{1}{e^{\beta (\epsilon - \mu)} - 1} \); Fermi-Dirac distribution: \( \bar{n}_{FD} = \frac{1}{e^{\beta (\epsilon - \mu)} + 1} \)

Fermi (3D): \( N = gV/(2\pi)^3 \int d^3k = \int \rho(\epsilon) d\epsilon \) to get \( k_F \) where \( \rho \) is density of state and \( g \) is degeneracy at each \( k \) (ex. spins, polarization). Change it accordingly for 2D

Sommerfeld expansion \( (k_BT \ll \epsilon_F) \): \( \int_0^\infty \frac{H(\epsilon)d\epsilon}{e^{\beta (\epsilon - \mu)} + 1} \approx \int_0^{\mu} H(\epsilon) d\epsilon + \frac{\pi^2}{6} (\frac{1}{\beta})^2 H'(\mu) + O\left(\frac{1}{\beta^3} \right) \)

4.6. Photon gas. blah
Plank distribution: \( \bar{n}_{pl} = \frac{1}{e^{\beta \epsilon} - 1} \) \( \)Plank spectrum: \( u(\epsilon) = \frac{8\pi}{(hc)^2} \frac{\epsilon^3}{e^{\beta \epsilon} - 1} \)

Capacity: \( C_V \sim T^3 \) \( \)Black body:

4.7. Debye model. blah
Use plank distribution, upper bound is similar to Fermi energy. Debye temperature: \( k_BT_D = h\omega_D \). When \( T \ll T_D \), \( C_V \sim (T/T_D)^3 \) for solid and \( C_V \sim \gamma T + (T/T_D)^3 \) for metal where \( \gamma T \) terms comes from degenerate electron Fermi gas.

4.8. Phase transition. blah
Critical T: \( \frac{\partial P}{\partial T} \bigg|_T = 0 \) and \( \frac{\partial^2 P}{\partial T^2} \bigg|_T = 0 \)

Clausius Clapeyron relation: \( \frac{\partial P}{\partial T} = \frac{L}{T \Delta V} \) where \( L \) is specific latent heat and \( \Delta v \) is change of specific volume
4.9. **BEC.** blah

\[ N = N_0 + N_e = \frac{x}{1-z} + \frac{V}{(2\pi)^7} \int \frac{d^3k}{z - \frac{1}{e^{\beta\mu} - 1}} \]

where \[ z = e^{\beta\mu} \]. When \( N_0 \) is sufficient large, \( z \to 1 \).

Critical temperature \( T_c \) is determined by \( N_e = N \).

4.10. **cycle.** blah

Carnot heat engine: \( e \equiv \frac{dW}{dQ_{in}} = 1 - T_c/T_h \) or just \( \frac{dQ_{in}}{dQ_{out}} = \frac{T_h}{T_c} \); Refrigerator: \( e = \frac{T_c}{T_h - T_c} \).

4.11. **Mean field approximation.** blah

Approximate \( H = -J \sum_{i,j} \sigma_i \sigma_{i,j} \) which sum over nearest N neighbor as \( H_m = -M \sum_i \sigma_i \) where \( M = NJ \langle \sigma_i \rangle \).

4.12. **Brownian motion.** blah

Langevin EOM: \( m \frac{dv(t)}{dt} + 6\pi\eta a v(t) = F(t) \), where \( F(t) \) is a stochastic variable which satisfies \( \langle F(t) \rangle = 0 \) and \( \langle F(t) \cdot F(t') \rangle = g\delta(t-t') \). \( \langle \ldots \rangle \) means average with respect to the distribution of the realizations of the stochastic variable.