On the Calculation of Full and Partial Directivity Indices

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Abstract

A set of metrics is presented which quantifies, using measured power spectra, the directivity of a loudspeaker, relative either to the full sphere or to a given hemisphere, plane, or half-plane. The former, called the *directivity index* (DI), is a commonly-used metric, but is only defined for complete sets of measurements. The latter metrics, called *partial directivity indices*, extend the definition of the DI to isolate sections of the loudspeaker's radiation pattern (e.g., forward radiation alone) and quantify its directivity over those sections. In general, all DI spectra are given by the ratio between the on-axis power spectrum and a weighted average of a certain subset of power spectra. For the spherical and hemispherical cases, each weight is proportional to the surface area of the portion of the unit sphere represented by the corresponding measurement, whereas for the planar cases, each weight is proportional to the arc length of the corresponding portion of the unit circle. In this document, the partial directivity indices are defined and the relevant formulae needed to compute them are provided.

1 Introduction

It is a well-known property of modern loudspeakers that sound is not radiated equally in all directions for all frequencies. The *directivity* of a loudspeaker describes the extent to which the acoustic power produced by the loudspeaker is biased toward a given direction. To quantify directivity, the *directivity factor* (or *directivity index* when expressed in dB) is defined as the acoustic intensity at a given point on the surface of an imaginary sphere surrounding the loudspeaker relative to the average intensity over the entire surface [1, 2]. The directivity factor (index) spectrum is typically defined as (ten times the base-ten logarithm of) the ratio between the on-axis power spectrum and the average power spectrum,[1] which is approximately computed by averaging measured power spectra over many directions [2]. In practice, rather than measuring power spectra uniformly distributed on the sphere (a time-consuming and complicated process), measurements are typically performed only along horizontal and vertical "orbits" around the loudspeaker [2]. We shall refer to the DI spectrum computed with these measurements as the *approximate full-sphere DI spectrum.*

In this document, we first recall the definitions of the exact and approximate full-sphere DI spectra. We then extend the latter to define three *partial* DI spectra:

- single plane (horizontal or vertical),
- single hemisphere (frontal or rear), and
- single half-plane (frontal or rear, and horizontal or vertical).

[1] This average power spectrum over many directions is sometimes called the *sound power* spectrum [2].

These partial DI spectra allow certain sections of the radiation pattern to be isolated when evaluating directivity. Of particular significance to these definitions are the weighted averages of measurements which yield the corresponding average power spectra. Consequently, we discuss the calculation of appropriate weights based on the geometrical arrangement of the measurements. We also discuss the calculation of the average and standard deviation of a given DI spectrum to serve as compact metrics by which to compare the directivities of different loudspeakers. This document is presented as part of the ongoing experimental survey of loudspeaker directivity at the 3D Audio and Applied Acoustics (3D3A) Laboratory at Princeton University.[2]

1.1 Coordinate System

In this document, we adopt a spherical coordinate system in accordance with the Audio Engineering Society (AES) standard AES56-2008 [3]. A diagram of this coordinate system is shown in Fig. 1. Typically, we define the *origin* of the coordinate system as the center of the loudspeaker cabinet, and a *reference axis* which passes through the center of the (upright) loudspeaker cabinet horizontally, parallel to the ground and perpendicular to a specified radiating surface (e.g., the dome of the high-frequency transducer). However, the precise alignment of the loudspeaker in the coordinate system may vary for different loudspeakers. We also define a *measurement axis* which passes through the origin and the measurement point (i.e., the position of the microphone). The plane defined by the reference axis and the measurement axis is called the *measurement plane*.

Let $\theta \in [0, \pi]$ denote the angle between the measurement axis and the reference axis on the measurement plane. Also let $\phi \in [0, 2\pi)$ denote the angle of the measurement plane about the reference axis, such that $\phi = 0$ when the measurement point is to the left of the loudspeaker (when viewed from the rear) and the measurement plane is parallel to the ground. When viewing the loudspeaker from the rear, ϕ increases as the measurement plane rotates clockwise about the reference axis. By this convention, measurements taken along the horizontal orbit will have ϕ values of either 0 (to the left of the loudspeaker) or π (to the right), and those along the vertical orbit will have ϕ values of either $\pi/2$ (above) or $3\pi/2$ (below). We will also refer to the corresponding Cartesian coordinate system, for which the x-axis points in the direction $(\pi/2, 0)$, the y-axis points in the direction $(\pi/2, \pi/2)$, and the z-axis points in the direction $(0, 0)$.

2 The Directivity Index

For measurements performed at a fixed distance away from the loudspeaker, let $H(\omega, \theta, \phi)$ denote the frequency response of the loudspeaker, where ω is angular frequency in rad/sec. We refer to the frequency response measured at $\theta = \phi = 0$ as the "on-axis" response, and denote it by $H_{0,0}(\omega) = H(\omega, 0, 0)$. The directivity index spectrum (in dB) is then given by

$$\mathrm{DI}(\omega) = 10 \log_{10} \frac{|H_{0,0}(\omega)|^2}{\frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} |H(\omega, \theta, \phi)|^2 \sin\theta d\theta d\phi}. \tag{1}$$

[2] `http://www.princeton.edu/3D3A/Directivity.html`

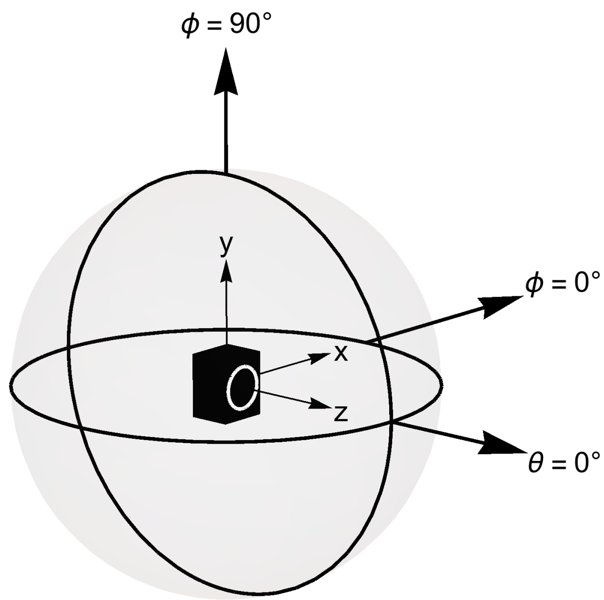


Figure 1: Diagram of the coordinate system. The reference axis ($\theta = 0$) passes through the center of the (upright) loudspeaker cabinet, parallel to the ground. The two circles represent the horizontal and vertical measurement orbits.

As it is impossible to measure H for all directions (θ, ϕ), we must turn to an approximate definition of the DI spectrum.

Consider a set of $2N$ directions $(\theta_{m,n}, \phi_{m,n})$ for $m = 0, 1$ and integer values of $n \in [0, N-1]$, where the value of m indicates the orbit and N is the number of measurements taken along each orbit. We denote frequency responses measured for these directions by $H_{m,n}(\omega) = H(\omega, \theta_{m,n}, \phi_{m,n})$. Let $\Delta\theta = 2\pi/N$ denote the angular spacing between measurements in each orbit, and $\Delta\phi = \pi/2$ denote the angular spacing between orbits. The full set of measurement directions are then given by

$$(\theta_{m,n}, \phi_{m,n}) = \begin{cases} (n\Delta\theta, m\Delta\phi) & \text{for } 0 \leq n < \frac{N}{2}, \\ \left(\pi - \left(n - \frac{N}{2}\right)\Delta\theta, \pi + m\Delta\phi\right) & \text{for } \frac{N}{2} \leq n < N, \end{cases} \tag{2}$$

where N is an integer multiple of 4 (i.e., the same number of measurements are taken in each quadrant of each orbit). By this definition, $m = 0$ corresponds to the horizontal orbit and $m = 1$ corresponds to the vertical orbit. Note that each orbit contains a measurement for the frontal on-axis ($\theta = 0$) response as well as the rear axis ($\theta = \pi$) response. For measurements taken in the 3D3A Lab, we maintain an angular spacing of $\Delta\theta = 5°$, yielding $N = 72$ total measurements on each orbit. A diagram illustrating an example of these measurement directions with an angular spacing of $\Delta\theta = 10°$ (for clarity) is shown in Fig. 4 (in the appendix).

Given frequency response measurements for each of the directions defined in Eq. (2), the approximate full-sphere DI spectrum (in dB) is given by

$$\mathrm{DI}(\omega) = 10\log_{10}\frac{\left|H_{0,0}(\omega)\right|^2}{\displaystyle\sum_{m=0}^{1}\sum_{n=0}^{N-1} w_n \left|H_{m,n}(\omega)\right|^2}, \tag{3}$$

where w_n is a direction-dependent weight attributed to each measurement. Note that these weights are normalized such that

$$\sum_{m=0}^{1}\sum_{n=0}^{N-1} w_n = 1.$$

This formula has been adapted from the method proposed by Davis [4], and later refined by Davis [5, 6] and Wilson [7, 8], in which measurements are equally spaced around each orbit, but weighted according to the surface area of the portion of the unit sphere represented by each measurement. This method will be discussed further in Section 4.

3 Partial Directivity Indices

In this section, we define the three types of partial directivity indices listed in Section 1.

3.1 Single Plane

The single plane DI uses measurements performed along a single orbit (either horizontal or vertical) to quantify the ratio between the on-axis acoustic output of the loudspeaker and its total acoustic output into the measurement plane. As defined by Eq. (2), these directions correspond to $n \in [0, N-1]$ and either $m = 0$ (horizontal) or $m = 1$ (vertical). A diagram illustrating an example of these measurement directions (in the horizontal plane) is shown in Fig. 5.

Given frequency response measurements for these directions, the approximate single-plane DI spectrum (in dB) is given by

$$\mathrm{DI}_m(\omega) = 10\log_{10}\frac{\left|H_{m,0}(\omega)\right|^2}{\displaystyle\frac{1}{N}\sum_{n=0}^{N-1} \left|H_{m,n}(\omega)\right|^2}. \tag{4}$$

Note that, by this definition, each measurement is given equal weight in the calculation of the average power spectrum. This is done because the measurements are equally spaced along the measurement orbit and, for this particular DI spectrum, we are interested only in the variation of the loudspeaker's frequency response with angle along that orbit.

3.2 Single Hemisphere

The single hemisphere DI uses measurements performed along halves (either frontal or rear) of both the horizontal and vertical orbits to quantify the ratio between the on-axis acoustic output of the loudspeaker and its total acoustic output into the measurement half-space (hemisphere). As defined by Eq. (2), these directions correspond to $m = 0, 1$ and either $n \in [0, N/4] \cup [3N/4, N-1]$ (frontal) or $n \in [N/4, 3N/4]$ (rear). Due to the inclusion of the end-points, an angular spacing of $\Delta\theta = 5°$ yields $N + 2 = 74$ total measurements. A diagram illustrating an example of these measurement directions is shown in Fig. 6.

Given frequency response measurements for these directions, the approximate single-hemisphere DI spectrum (in dB) is given by

$$\mathrm{DI}(\omega) = 10 \log_{10} \frac{\left|H_{0,0}\left(\omega\right)\right|^2}{\displaystyle\sum_{m=0}^{1} \sum_{n=0}^{N-1} v_n \left|H_{m,n}\left(\omega\right)\right|^2}, \tag{5}$$

where the weights v_n in the above equation are related to those used in the definition of the full-sphere DI spectrum (see Eq. (3)) by, in the case of the frontal hemisphere,

$$v_n = \begin{cases} 2w_n & \text{for } n \in [0, N/4) \cup (3N/4, N-1], \\ w_n & \text{for } n = N/4, 3N/4, \\ 0 & \text{for } n \in (N/4, 3N/4). \end{cases} \tag{6}$$

In the case of the rear hemisphere, the weight values for the first and third cases in Eq. (6) are swapped. Due to the symmetry of the w_n weights (as will be seen in Section 4), the v_n weights are also normalized such that

$$\sum_{m=0}^{1} \sum_{n=0}^{N-1} v_n = 1.$$

Note that the weights at the end-points of each half-orbit (i.e., the four measurements for which $\theta = \pi/2$) are reduced by a factor of 2 relative to the other non-zero weights. This is necessary since the total weight attributed to those measurements in the calculation of the full-sphere DI spectrum (Eq. (3)) represents a portion of the sphere whose surface area is evenly split in front of and behind the loudspeaker, i.e., between $\theta = \pi/2 \pm \Delta\theta/2$. Consequently, to compute the average power spectrum for the frontal hemisphere only, for example, we include only the forward-radiating power by splitting the surface area (and therefore the weight) in half.

3.3 Single Half-Plane

The single half-plane DI uses measurements performed along half (either frontal or rear) of a single orbit (either horizontal or vertical) to quantify the ratio between the on-axis acoustic output of the loudspeaker and its total acoustic output into the measurement half-plane. As defined by Eq. (2), these directions correspond to any combination of $m = 0$ (horizontal) or $m = 1$ (vertical) and $n \in [0, N/4] \cup [3N/4, N-1]$ (frontal) or $n \in [N/4, 3N/4]$ (rear). Due to the inclusion of the end-points, an angular spacing of $\Delta\theta = 5°$ yields $N/2 + 1 = 37$ total measurements. A diagram illustrating an example of these measurement directions is shown in Fig. 7.

Given frequency response measurements for these directions, the approximate single-half-plane DI spectrum (in dB) is given by

$$\mathrm{DI}_m(\omega) = 10\log_{10}\frac{|H_{0,0}(\omega)|^2}{\dfrac{1}{N/2}\displaystyle\sum_{n=0}^{N-1} u_n\,|H_{m,n}(\omega)|^2}, \tag{7}$$

where the weights u_n in the above equation are given by, in the case of a frontal half-plane,

$$u_n = \begin{cases} 1 & \text{for } n \in [0, N/4) \cup (3N/4, N-1], \\ \frac{1}{2} & \text{for } n = N/4, 3N/4, \\ 0 & \text{for } n \in (N/4, 3N/4). \end{cases} \tag{8}$$

In the case of a rear half-plane, the weight values for the first and third cases in Eq. (8) are swapped. It is easily shown that these weights satisfy

$$\sum_{n=0}^{N-1} u_n = N/2.$$

Note that, as in the single-plane DI spectrum definition given in Eq. (4), all measurements, with the exception of those at the end-points, are given equal weight in the calculation of the average power spectrum. However, as in the single-hemisphere DI spectrum definition given in Eqs. (5) and (6), the weights of the end-point measurements are reduced by a factor of 2.

4 Weight Calculations

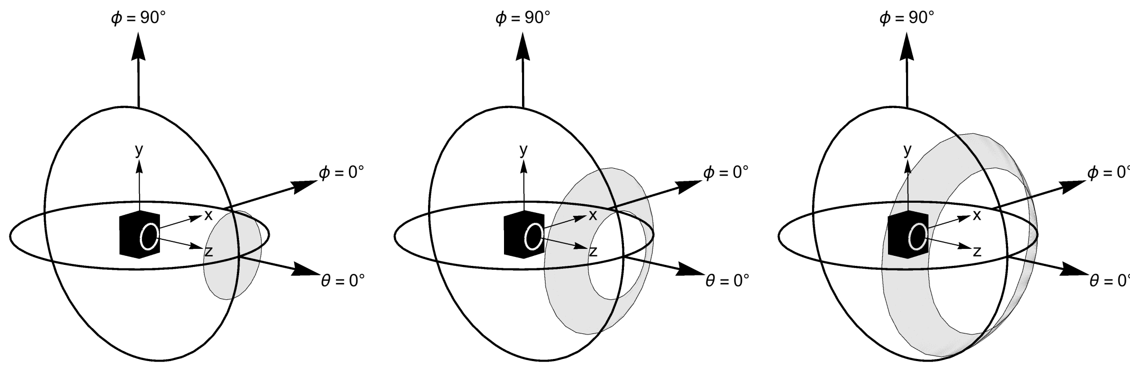
As mentioned above, in order to compute the full-sphere and single-hemisphere DI spectra, we must compute a weighted average of frequency response measurements. In this section, we describe how the appropriate weights are calculated. This method follows the same formulation as that proposed by Wilson [7, 8], which itself is equivalent to the graphical method proposed by Kendig and Mueser [9].

Consider a plane perpendicular to the z-axis at some distance $z \in [-1, 1]$ from the origin. The intersection of this plane and the unit sphere defines a circle (or a point if $|z| = 1$) consisting of all directions which satisfy $z = \cos\theta$. The surface area of the portion of the unit sphere (i.e., the solid angle) for which $z \geq \cos\theta$ is given by

$$\Omega(\theta) = \int_{\phi=0}^{2\pi}\int_{\theta'=0}^{\theta} \sin\theta' d\theta' d\phi = 2\pi(1-\cos\theta), \tag{9}$$

where θ' is a variable of integration and Ω is given in sr.

We use the above equation to compute weights that correspond to the surface area of the unit sphere represented by each measurement. For example, the surface area represented by an on-axis measurement (for which $\theta = 0$) is given by $\Omega(\Delta\theta/2)$. This surface can be visualized as a "spherical



(a) w_0 (b) w_1 (c) w_2

Figure 2: Surface areas corresponding to various weight calculations.

cap," as illustrated in Fig. 2a. In the case of measurements along both the horizontal and vertical orbits, the total surface area of the unit sphere represented by the four measurements of the same θ is given by $\Omega(\theta + \Delta\theta/2) - \Omega(\theta - \Delta\theta/2)$. Such a surface can be visualized as a "strip" around the sphere, as illustrated in Figs. 2b and 2c.

Following the numbering convention defined by Eq. (2), the first few weights for the calculation of the average power spectrum are given by

$$w_0 = \frac{1}{4\pi}\frac{\Omega\left(\frac{\Delta\theta}{2}\right)}{2}, \tag{10}$$

$$w_1 = \frac{1}{4\pi}\frac{\Omega\left(\Delta\theta + \frac{\Delta\theta}{2}\right) - \Omega\left(\Delta\theta - \frac{\Delta\theta}{2}\right)}{4}, \text{ and} \tag{11}$$

$$w_2 = \frac{1}{4\pi}\frac{\Omega\left(2\Delta\theta + \frac{\Delta\theta}{2}\right) - \Omega\left(2\Delta\theta - \frac{\Delta\theta}{2}\right)}{4}. \tag{12}$$

In the above equations, the on-axis weight w_0 is divided by 2 since the surface area of the spherical cap must be evenly split between the on-axis measurement from the horizontal orbit and that from the vertical orbit. Similarly, both the weights w_1 and w_2 are divided by 4 since the surface area of the strip around the sphere must be evenly split between the two measurements from the horizontal orbit and those from the vertical orbit. The remaining weights for a given set of measurements may be calculated by extending the pattern established by the above equations. Figure 3 shows the full set of weights for the calculation of the approximate full-sphere DI spectrum with an angular spacing between measurements of $\Delta\theta = 5°$. A certain subset of these weights is applied in the calculation of the approximate single-hemisphere DI spectrum, as given by Eq. (6).

It is useful to note that the full set of weights will exhibit a certain periodicity, as seen in Fig. 3, due to the symmetry of the measurements along each orbit. Indeed, only the first $N/4+1$ weights for the calculation of the approximate full-sphere DI spectrum need to be computed explicitly. These values are tabulated in Table 1 for an angular spacing of $\Delta\theta = 5°$.

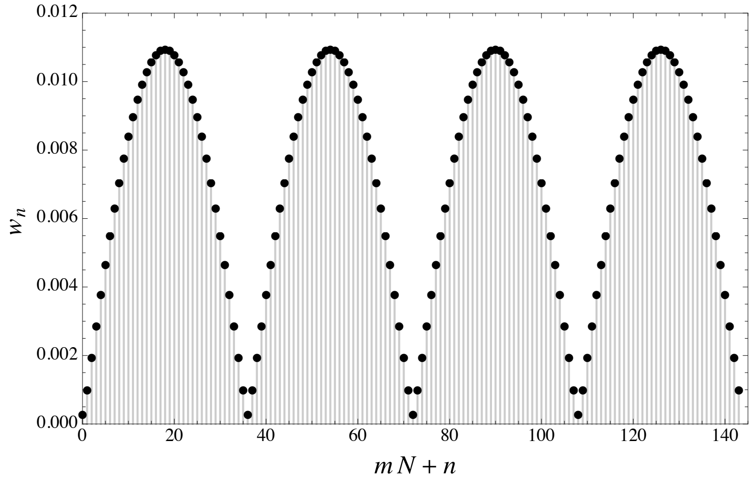


Figure 3: Full-sphere DI spectrum weights for $\Delta\theta = 5°$. The first $N = 72$ weights correspond to the horizontal plane ($m = 0$), while the second correspond to the vertical ($m = 1$).

n	θ	w_n
0	0°	0.000237945
1	5°	0.00095042
2	10°	0.00189361
3	15°	0.00282238
4	20°	0.00372968
5	25°	0.00460859
6	30°	0.00545242
7	35°	0.00625476
8	40°	0.0070095
9	45°	0.00771089
10	50°	0.0083536
11	55°	0.00893273
12	60°	0.00944387
13	65°	0.00988315
14	70°	0.0102472
15	75°	0.0105333
16	80°	0.0107392
17	85°	0.0108634
18	90°	0.0109048

Table 1: Full-sphere DI spectrum weights for $\Delta\theta = 5°$. This corresponds to a total of $N = 72$ measurements on each orbit. Only the first $N/4+1 = 19$ values are tabulated due to the periodicity of w_n, as shown in Fig. 3.

5 Average Directivity Index

In order to more easily compare the directivity of different loudspeakers, we now define an average DI value. This value is computed by averaging the DI spectrum over a certain range of frequencies. For measurements taken in the 3D3A Lab, we compute a logarithmically-weighted average of each approximate DI spectrum from 100 Hz to 20 kHz.

In practice, the frequency responses used in the various DI spectra definitions given in previous sections are computed via the fast Fourier transform (FFT). Therefore, the discrete frequencies over which any frequency spectrum is defined are uniformly spaced between 0 Hz (DC) and the sampling frequency. The frequency index, $k \in [0, K-1]$, is proportional to the actual frequency in Hz, given by the relationship

$$f[k] = \frac{F_s k}{K}, \tag{13}$$

where K is the number of FFT points and F_s is the sampling frequency.

Rather than computing the arithmetic mean of the DI spectrum on a linear frequency scale, we compute a weighted average which approximates averaging the spectrum over a logarithmic frequency scale. The average DI spectrum value (in dB) over the range $k \in [k_1, k_2]$ is then given by

$$\overline{\mathrm{DI}} = \frac{\displaystyle\sum_{k=k_1}^{k_2} W[k] \cdot \mathrm{DI}[k]}{\displaystyle\sum_{k=k_1}^{k_2} W[k]}, \tag{14}$$

where $k_1 < k_2 \leq K/2$ and $W[k]$ is a sequence of weights given by

$$W[k] = \log \frac{k + 0.5}{k - 0.5}. \tag{15}$$

Similarly, the variance of the DI spectrum over the same frequency range is given by

$$\sigma^2_{\mathrm{DI}} = \frac{\displaystyle\sum_{k=k_1}^{k_2} W[k] \left(\mathrm{DI}[k] - \overline{\mathrm{DI}}\right)^2}{\displaystyle\sum_{k=k_1}^{k_2} W[k]}. \tag{16}$$

The standard deviation, σ_{DI}, may be used to quantify the extent to which the loudspeaker exhibits *constant directivity*, as a small standard deviation indicates that the DI is constant with frequency. This, however, does not *guarantee* constant directivity, as a small σ_{DI} means only that the ratio between the on-axis and the average power spectrum remains nearly constant with frequency – the polar radiation pattern may still vary with frequency. Additional metrics for constant directivity will be discussed in a future publication.

References

[1] C. T. Molloy. Calculation of the Directivity Index for Various Types of Radiators. *The Journal of the Acoustical Society of America*, 20(4):387–405, July 1943.

[2] Floyd E. Toole. *Sound Reproduction: Loudspeakers and Rooms*. Focal Press, 2008.

[3] AES56-2008: AES standard on acoustics - Sound source modeling - Loudspeaker polar radiation measurements, 2008 (reaffirmed 2014).

[4] Don Davis. A Proposed Standard Method of Measuring the Directivity Factor "Q" of Loudspeakers Used in Commercial Sound Work. *J. Audio Eng. Soc.*, 21(7):571–578, 1973. URL `http://www.aes.org/e-lib/browse.cfm?elib=1948`.

[5] Don Davis. On Standardizing the Measurement of Q. *J. Audio Eng. Soc.* (Forum), 21(9): 730–731, 1973.

[6] Don Davis. Further Comments on Directivity Factor. *J. Audio Eng. Soc.* (Forum), 21(10): 827–828, 1973. URL `http://www.aes.org/e-lib/browse.cfm?elib=1917`.

[7] Geoffrey L. Wilson. Directivity Factor: Q or R_θ? Standard Terminology and Measurement Methods. *J. Audio Eng. Soc.* (Forum), 21(10):828, 830, 833, 1973. URL `http://www.aes.org/e-lib/browse.cfm?elib=10298`.

[8] Geoffrey L. Wilson. More on the Measurement of the Directivity Factor. *J. Audio Eng. Soc.* (Forum), 22(3):180, 182, 1974. URL `http://www.aes.org/e-lib/browse.cfm?elib=2771`.

[9] Paul M. Kendig and Roland E. Mueser. A Simplified Method for Determining Transducer Directivity Index. *The Journal of the Acoustical Society of America*, 19(4):691–694, 1947. doi: http://dx.doi.org/10.1121/1.1916539. URL `http://scitation.aip.org/content/asa/journal/jasa/19/4/10.1121/1.1916539`.

Appendix: Measurement Diagrams

In this section, we present diagrams representing the measurement positions for each of the four cases described in Sections 2 and 3. For clarity, we use an angular spacing between measurements of $\Delta\theta = \pi/18$ rad $= 10^\circ$ in all diagrams. Figure 4 shows measurements for the full-sphere DI, consisting of complete horizontal and vertical orbits. Figure 5 shows measurements for the single-plane DI, consisting of a complete horizontal orbit only. Figure 6 shows measurements for the single-hemisphere DI, consisting of frontal halves only of both the horizontal and vertical orbits. Figure 7 shows measurements for the single-half-plane DI, consisting of the frontal half of the horizontal orbit only.

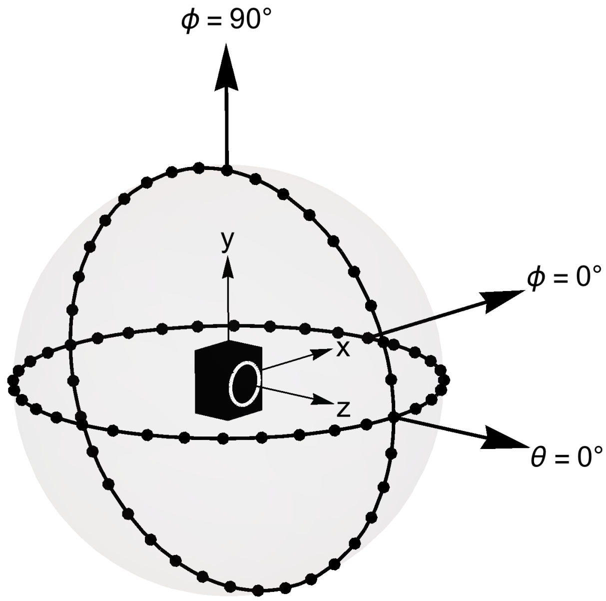


Figure 4: Diagram of full-sphere measurements.

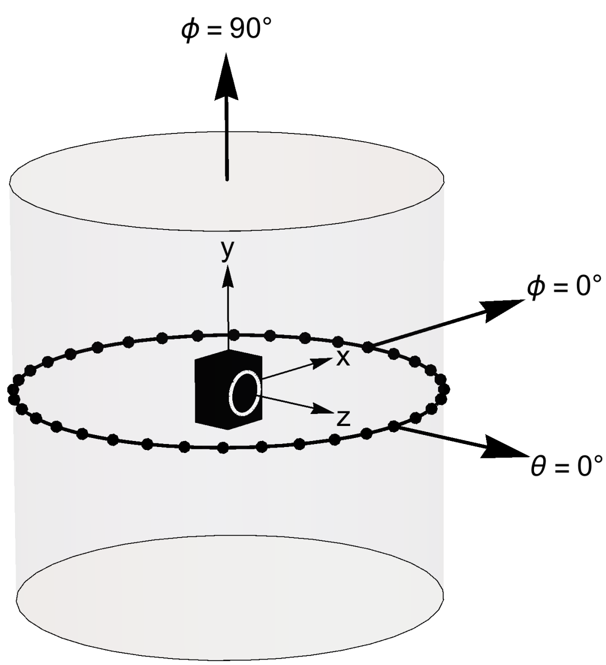


Figure 5: Diagram of single-plane measurements.

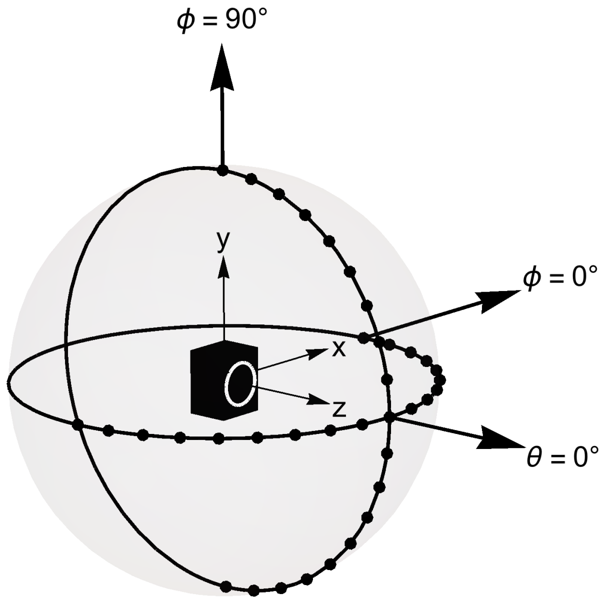


Figure 6: Diagram of single-hemisphere measurements.

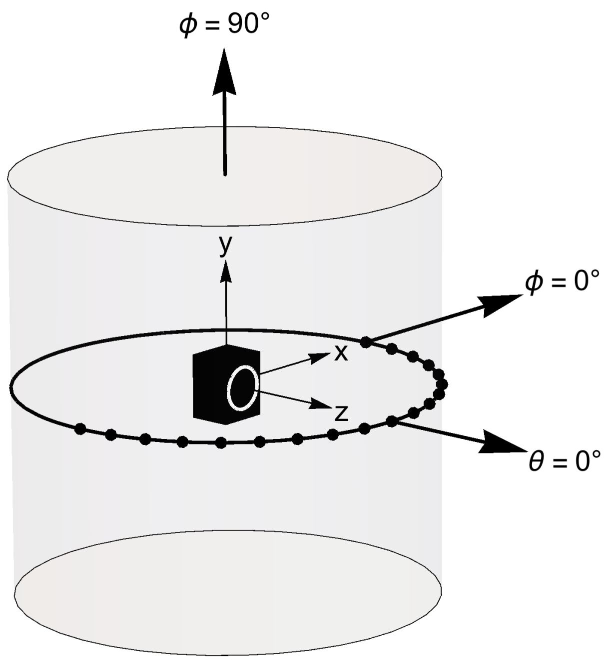


Figure 7: Diagram of single-half-plane measurements.