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A modified kinetic theory for frictional granular flows in dense and dilute regimes

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Continuum modeling of granular and gas-solid flows generally involves the use of a kinetic-theory (KT) model for the particulate phase, and the most widely used KT models have been derived for dilute flows of smooth, frictionless spheres. In reality, however, granular particles are frictional and can achieve dense packing, and these features must be taken into account to improve rheological predictions in these flow scenarios. Existing approaches in the literature for producing closed-form KT-based models employ empirical modifications to adapt the original models for use in dense and frictional systems. In this article, we investigate the capacity for such modifications to improve the rheological predictions of the Garzó–Dufty (GD) KT model [V. Garzó and J. W. Dufty, “Dense fluid transport for inelastic hard spheres,” Phys. Rev. E 59, 5895–5911 (1999)]. On the basis of molecular dynamics simulations of homogeneous, simple shear flows of soft, frictional spheres, we propose a new expression for the radial distribution function at contact as well as modifications to the GD expressions for shear stress and energy dissipation rate. These changes account for dense-regime scalings observed in inertial-number models as well as the effects of interparticle friction while preserving the dynamic nature of the KT model.

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I. INTRODUCTION

Granular rheology has been a commonly studied topic for many years owing in part to the diversity of behavior that granular flows exhibit. Depending on the local solids volume fraction \( \phi \) and shear rate \( \dot{\gamma} \), one can observe any of three distinct flow regimes:\cite{1,2,3} (1) a quasi-static regime for \( \phi > \phi_c \), in which pressure \( p \) and shear stress \( \tau \) are independent of shear rate; (2) an inertial regime below \( \phi_c \), in which \( p, \tau \sim \dot{\gamma}^2 \); and (3) an intermediate regime occurring in a narrow window of volume fractions about \( \phi_c \) in which \( p, \tau \sim \dot{\gamma}^n \) with \( 0.5 \leq n \leq 1 \). As particle stiffness \( k \to \infty \), only the inertial regime can be observed, and, because granular particles are generally extremely hard, a large subset of granular flows belong to the inertial regime.

Stresses in the inertial regime are typically modeled using the kinetic theory (KT) of granular gases,\cite{4,5,6,7} a class of dynamic models capable of predicting a variety of clustering behaviors.\cite{8,9} Development of these models involves following what we call the KT approach or KT analysis, which is the general process of evaluating the transport coefficients by moment analysis of the Boltzmann equation while making simplifying assumptions regarding the collision integral. These assumptions, which include the two-particle interaction model and the nature of the velocity distributions, will affect not only the regimes of validity for any resulting relations but also their attainability in closed form, as oftentimes the collision integral can only be evaluated numerically. The earliest and most widely used KT models have been derived for dilute flows of inelastic, smooth, frictionless spheres,\cite{4,5,6,7} though there also exist more sophisticated models that account for particle roughness by including tangential restitution and rotational degrees of freedom.\cite{10,11} We refer collectively to these two types of models, which consist of closed-form constitutive expressions, as traditional KT models.
Though all flows in the inertial regime are amenable to KT analysis, this regime in itself contains two ranges of volume fractions exhibiting somewhat different behavior. The first is the *dilute inertial* regime and encompasses volume fractions between zero and $\phi_f \approx 0.49$. In this range, traditional KT models exhibit generally good agreement with discrete particle simulations. On the other hand, for volume fractions between $\phi_f$ and $\phi_c$, which constitute the *dense inertial* regime, KT models are less successful, with discrepancies resulting from a few assumptions made in their derivation. For example, one typically assumes “molecular chaos,” i.e., that the collisions of a particle with its neighbors are uncorrelated, allowing one to express the two-particle velocity distribution function as the product of the single-particle distribution functions (which are Gaussian). In dense flows, though, a particle will interact many times with the same neighbors as steric effects become predominant, thereby resulting in a non-Gaussian distribution of normal relative velocities. Additionally, these models assume a small Péclet number $Pe \equiv \dot{\gamma}d/\sqrt{T}$, which represents the extent of departure from the equilibrium, isotropic state of an assembly of particles of diameter $d$ and with a granular temperature $T$. If $Pe \ll 1$, thermal forces predominate and preserve a nearly isotropic structure within the assembly. However, shear flows are distinctly nonequilibrium, as demonstrated by the emergence of prominent structural features including force chains and anisotropy of the fabric of contacts, which would not survive in a thermal system. This phenomenon arises in part because collisions between granular particles are dissipative, characterized by a restitution coefficient $\epsilon$ often not near unity that serves to increase $Pe$ by decreasing $T$. It is hence understandable that traditional KT models are unable to predict some features of the dense inertial regime, including the increase in $T$ near the critical volume fraction and the saturation of the shear stress ratio $\eta \equiv \tau/p$ to its correct close-packing limit. Recent inertial-number models do predict the yield stress but do not apply to dilute flows; they also do not account for important nonlocal effects that KT can capture via conduction of pseudothermal energy. There is a need, therefore, for a rheological model that bridges the dense and dilute behaviors within the KT framework.

In addition to their dense-regime limitations, traditional KT models are derived assuming collisions of *frictionless* particles only and hence are unequipped to handle any dependence on the interparticle friction coefficient $\mu$. In reality, however, granular materials are frictional, and the value of $\mu$ has been shown to have a significant effect on stresses at least in the quasistatic regime, where increasing $\mu$ generally increases the stress because of larger tangential interparticle forces arising from enduring contacts. Attempts to quantify the friction effect in the inertial regime, though, have been somewhat more limited. Many KT works have considered tangential restitution, which similarly describes roughness but is nevertheless microscopically different from friction. A KT model that explicitly and accurately accounts for dependence on $\mu$ is currently lacking.

One important quantity in which friction, shearing, and large packing density all play an important role is the radial distribution function at contact $g_0(\phi)$. Most expressions for $g_0$, in agreement with $g_0$ data extrapolated directly from equilibrium hard-sphere simulations, predict its divergence at close packing. i.e., $g_0 \sim (\phi_c - \phi)^m$ for some constant $m < 0$. However, shear simulations of discrete-particle assemblies suggest that these expressions significantly underpredict $g_0$ for systems far from equilibrium and featuring strong contact anisotropy. Additionally, these expressions tend to neglect the dependence of the critical volume fraction on the friction coefficient, which has been observed both in static systems and sheared systems. Since $\phi_c(\mu)$ determines the point at which the pressure diverges, any extension of KT to the dense regime must take this dependence into account.

Given the need for a KT model that spans dense and dilute regimes while accounting for anisotropy and interparticle friction, it is natural to attempt to revisit the KT approach without making the aforementioned inappropriate assumptions. Doing so would, in theory, yield a model applicable to a wide array of flows and particle types without the need for empirical fitting of rheological data. However, two major problems arise in this endeavor. First, at the present time it does not appear possible to deduce closed-form expressions for the transport coefficients in the dense regime when employing the non-Gaussian relative velocity distribution function. Because the rheological behavior of the material would need to be evaluated numerically for every given set of particle parameters, it is impractical to implement such a model into a continuum-model solver for application to large-scale flow problems. Second, this approach still does not avoid the need for
empiricism, as the dense-regime relative velocity distribution function and the radial distribution function at contact cannot yet be predicted a priori. Hence, it appears that the most promising path toward a comprehensive rheological model in the near term lies in empirical modification of existing closed-form KT models.

Indeed, such KT modifications have been proposed numerous times in the recent literature. Of particular interest are the changes proposed by Jenkins and co-workers for extending the commonly used theory of Garzó and Dufty (GD), which is originally derived for flows of frictionless, moderately dissipative spheres in the dilute regime. The first modification calls for introduction of a length scale $L$ which the authors call a “chain length,” into the expression for the energy dissipation rate. This length scale is said to represent the typical length of force chains that form in dense granular flows and that grow large upon approaching $\phi_c$. While the existence of a large or diverging length scale in such flows is a topic of debate, the use of such a functional form is nevertheless successful in qualitatively describing the dense-regime increase in temperature and decrease in shear stress ratio. The second modification is the replacement of the normal restitution coefficient with an “effective” restitution coefficient $e_{\text{eff}}(\mu, e) < e$ that accounts for the increased dissipation resulting from interparticle friction. This substitution has been shown to improve stress predictions for dilute flows of slightly frictional particles. Modifications such as these are not only successful in improving the predictions of traditional KT models but moreover are convenient to constitute based on stress and temperature data from granular flow experiments and simulations.

In the present work, we assess the capacity for these two modifications to yield accurate model expressions by comparing their steady-state predictions for pressure, shear stress, and temperature with data obtained from discrete element method (DEM) simulations of sheared particle assemblies; and, when these approaches prove insufficient, we propose additional corrections to bring the GD model predictions into closer agreement with those of the DEM simulations.

In Secs. II–IV, we present the original (GD) KT model, an explanation of our simulation methodology, and a validation of our simulations for the case of frictionless particles below $\phi_f$. We then demonstrate the stark disagreement between the GD model predictions and our DEM results above $\phi_f$ and for frictional particles at all volume fractions. Finally, we outline the steps taken to constitute various model quantities used in our modified KT expressions based on our DEM results.

II. GARZÓ–DUFFY KT MODEL

As aforementioned, the kinetic theory of choice for the modifications will be that of Garzó and Dufty, which is designed for smooth, frictionless, moderately inelastic particles in the dilute regime. According to this model, for simple shear flows one can write the pressure $p$ as

$$p = \rho_s H(\phi, e)T,$$

the shear stress $\tau$ as

$$\tau = \rho_s d\dot{\gamma} J(\phi, e)\sqrt{T},$$

and the energy dissipation rate $\Gamma$ as

$$\Gamma = \frac{\rho_s}{d} K(\phi, e)T^{3/2}.$$

Here, $\rho_s$ is the solid material density, and the definitions of the dimensionless expressions $H, J,$ and $K$ and their components are

$$H(\phi, e) = \phi[1 + 2(1 + e)\phi g_0],$$

$$J(\phi, e) = \frac{5\sqrt{\pi}}{96} \eta^*,$$

$$K(\phi, e) = \frac{12}{\sqrt{\pi}} \phi^2 g_0(1 - e^2),$$

$$\eta^* = \eta_L^* + \eta_c^* + \eta_b^*.$$
\[ \eta_k^* = \frac{1 - \frac{3}{5}(1 + e)(1 - 3e)\phi g_0}{[1 - \frac{1}{4}(1 - e)^2 - \frac{5}{24}(1 - e^2)]g_0}, \]  
(8)

\[ \eta_c^* = \frac{4}{3}(1 + e)\phi g_0 \eta_k^*, \]  
(9)

and

\[ \eta_b^* = \frac{384}{25\pi}(1 + e)\phi^2 g_0. \]  
(10)

The quantities \( \eta_k^* \), \( \eta_c^* \), and \( \eta_b^* \) are the scaled kinetic, collisional, and bulk viscosity contributions, respectively. As per Jenkins and Berzi,\(^31\) we have neglected here the very small contribution of all terms proportional to a quantity that Garzó and Dufty\(^4\) denote as \( c^*(e) \). The radial distribution function at contact \( g_0(\phi) \) has been modeled in numerous ways,\(^{26-28,32,38}\) but two formulations in particular have been validated by simulation data of frictionless hard spheres at equilibrium. The first of these is the Carnahan–Starling expression,\(^38\)

\[ g_{CS}^0 = 1 - \frac{\phi}{2}(1 - \phi)^3, \]  
(11)

which is valid for systems below \( \phi_f \). The second is a modification to \( g_{CS}^0 \) proposed by Torquato\(^{26}\) for denser systems. Written as

\[ g_{Torquato}^0 = \begin{cases} 
    g_{CS}^0(\phi) & \text{for } \phi \leq \phi_f \\
    g_{CS}^0(\phi_f) \frac{\phi - \phi_f}{\phi_c - \phi_f} & \text{for } \phi > \phi_f 
\end{cases}, \]  
(12)

it differs from the Carnahan–Starling result only above \( \phi_f \), specifically by diverging at \( \phi_c \). Because of its agreement with simulation data and its use in a recent KT work,\(^31\) we will initially use Torquato’s\(^{26}\) \( g_0 \) formulation in Sec. IV together with the GD equations for comparison with DEM results.

For steady-state simple shear flows, we can write the granular energy balance as

\[ \Gamma - J_{vis} = 0, \]  
(13)

where

\[ J_{vis} = \dot{\gamma} \tau \]  
(14)

is the viscous energy production rate. With this condition and Eqs. (2) and (3), we can solve for the GD steady-state temperature expression

\[ T_{GD}^{SS} = \left( \frac{J}{K} \right) (\dot{\gamma} d)^2. \]  
(15)

This result then allows us to obtain the steady-state GD pressure

\[ p_{GD}^{SS} = \left( \frac{JH}{K} \right) \rho_s (\dot{\gamma} d)^2 \]  
(16)

and shear stress

\[ \tau_{GD}^{SS} = \sqrt{\frac{J^3}{K} \rho_s (\dot{\gamma} d)^2}. \]  
(17)

Finally, the shear stress ratio \( \eta \) is found from Eqs. (1) and (2) to be

\[ \eta_{GD}^{SS} = \frac{\sqrt{JK}}{H}. \]  
(18)

We emphasize that these four expressions are valid only for simple shear flows that have reached steady state.
III. SIMULATION METHODS

We use a package of the discrete element method\(^3\) found in the molecular dynamics package LAMMPS\(^3\) in order to simulate simple shear flows of discrete, uniform, spherical particles. These particles interact via a linear spring-dashpot (LSD) model, which gives the normal and tangential forces on a particle \(i\) caused by contact with a particle \(j\) as

\[
F_{ij}^n = k_n \delta_{ij} n_i - \gamma_n m_{eff} v_{ij}^n, \quad (19)
\]

\[
F_{ij}^t = -k_t u_{ij}^t - \gamma_t m_{eff} v_{ij}^t, \quad (20)
\]

for overlap distance \(\delta_{ij}\), particle diameter \(d\), spring stiffness constants \(k_n\) and \(k_t\), viscous damping constants \(\gamma_n\) and \(\gamma_t\), effective mass \(m_{eff} = m_i m_j / (m_i + m_j)\) for particle masses \(m_i\) and \(m_j\), relative particle velocity components \(v_{ij}^n\) and \(v_{ij}^t\), and elastic shear displacement \(u_{ij}^t\). We choose to set \(k_t/k_n = 2/7\) and \(\gamma_t = 0\) as done in some previous works\(^4\) and by doing so can set \(\gamma_n\) in order to produce the desired restitution coefficient according to the expression

\[
e = \exp\left(-\gamma_n \pi / \sqrt{4k_n/m_{eff} - \gamma_n^2}\right).
\]

While experiments have revealed a dependence of the restitution coefficient on the particle impact velocity,\(^4\) a behavior captured by some nonlinear spring-dashpot models,\(^4\) kinetic theory models assume a constant \(e\) as produced by the LSD model. Hence, for ease of comparison we neglect more complex, nonlinear behavior.

Samples are prepared by placing 2000 particles on a simple cubic lattice in a periodic box, assigning random, normally distributed initial velocities to the particles, and allowing the assembly to evolve in order to lose any memory of the initial configuration. The assembly then undergoes a simple shear deformation with shear rate \(\dot{\gamma}\) imposed via the Lees-Edwards boundary condition,\(^4\) which allows for the system to remain homogeneous during the shearing process. The shearing motion proceeds until the system reaches a steady state and remains there for sufficient time to collect proper statistics as determined by the saturation of the time-averaged stresses. From the particle interaction forces and the constant box volume \(V\), we can calculate the macroscopic stress tensor as

\[
\sigma = \frac{1}{V} \sum_i\left[ \sum_{j \neq i} \frac{1}{2} \mathbf{r}_{ij} F_{ij} + m_i \mathbf{v}_i \right], \quad (21)
\]

where \(\mathbf{r}_{ij}\) is the contact vector from the center of particle \(j\) to the center of particle \(i\), and \(\mathbf{v}_i\) is the particle fluctuating velocity (i.e., its velocity relative to its mean streaming velocity). From this tensor, we can calculate the ensemble-averaged pressure \(p = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3\) and shear stress \(\tau = \sigma_{xx}\). Because we limit the focus of our study to inertial regime flows, i.e., flows with \(\phi < \phi_c\) and \(\dot{\gamma} \equiv \dot{\gamma} / \sqrt{k / \rho_s d} < 10^{-3}\) [Ref. 1], it is appropriate to scale our stress data by the quantity \(\rho_s (\dot{\gamma} d)^2\), where \(\rho_s\) is the solid material density and \(d\) is the particle diameter; similarly, we scale the temperature data by \((\dot{\gamma} d)^2\). We also have confirmed this Bagnold scaling by running a small set of simulations with varying values of \(k\) and observing no substantial changes in the predicted temperature or stresses.

IV. RESULTS

A. Comparison of KT with DEM of frictionless particles

If the kinetic theory is to be extended to frictional and dense flows based on DEM data, we must demonstrate first that the continuum and discrete models make similar stress and temperature predictions for frictionless particle flows at low-to-moderate volume fractions. Such a comparison is shown in Fig. 1, with scaled pressure, shear stress ratio, and scaled temperature plotted versus volume fraction.

Indeed, as expected, the predictions of the two models display good agreement up to \(\phi \approx \phi_c\). For higher volume fractions, however, there appear three salient discrepancies: (1) the scaling of \(p\) with respect to \(\phi\), (2) the close-packed limit of the shear stress ratio, and (3) the scaling of \(T\) with respect
FIG. 1. Results from DEM simulations of steady state simple shear flows of frictionless particles. (a) Scaled pressure, (b) shear stress ratio, and (c) scaled temperature are plotted versus volume fraction for various cases of the restitution coefficient $e$. The predictions of GD kinetic theory with Torquato’s radial distribution function and $\phi_c = 0.636$ are shown with solid lines. GD and DEM are in good agreement for volume fractions of $\phi \lesssim \phi_f = 0.49$ but depart above this point.

B. Comparison of KT with DEM of frictional particles

When the particles are made frictional, on the other hand, the DEM data depart markedly from the GD predictions, as seen in Fig. 2 for $\mu = 0.5$. In addition to its aforementioned dense-regime shortcomings, the GD model now substantially overpredicts the pressure and temperature for $\phi \lesssim \phi_f$ while also underpredicting the shear stress ratio. This behavior is unsurprising, as the GD model is not designed for such systems, and is explained by the additional source of dissipation of pseudothermal energy provided by interparticle friction during collisions between particles. Additionally, the assumption that the critical volume fraction $\phi_c$ is independent of $\mu$ leads the model to predict incorrectly the location of the dense-limit pressure divergence. As expected based on its derivation, the GD model is unequipped to handle either friction or dense packing, and adaptations must be made to extend its applicability to these conditions.

Based on the discrepancies observed between the GD and DEM predictions and on previous approaches to constitutive model refinement, our strategy for modifying the kinetic theory equations will be as follows:

1. The energy dissipation $\Gamma$ (Eq. (3)) will be modified to include (1) an effective restitution coefficient $e_{\text{eff}} = e_{\text{eff}}(e, \mu)$ in the $K$ term to capture $\mu$-functionality and (2) a multiplicative...
correction factor $\delta_T$ related to the chain length\textsuperscript{31} to capture dense-regime trends with respect to $\phi$.

2. A new expression will be proposed for the radial distribution function at contact $g_0$ that will diverge at close packing in a manner concordant with our DEM data and that is specifically for use in nonequilibrium systems.

3. The shear stress $\tau$ will be decomposed into a yield stress and an inertial stress, the latter of which will be similar to Eq. (2) except augmented with a multiplicative correction factor $\delta_T$ (analogous to the adjustable parameter that appears in the $\tau$ expression of some previous kinetic theories\textsuperscript{7,8,44}). This factor will transition from a dilute-regime limit of $\delta_T^{\text{dil}}$ that accounts for $\mu$-functionality at low $\phi$ to a dense-regime limit $\delta_T^{\text{dense}}$ that will force $\eta$ to obey an inertial-number model\textsuperscript{1,20–22} for $\phi > \phi_f$.

The correction factors will be applied to the dynamic GD equations and, as a result, will change the steady-shear GD predictions produced by imposing Eq. (13). The expressions chosen for these corrections, therefore, will be motivated by the steady-state values of the temperature, pressure, and shear stress calculated from our simple shear simulations.

C. Constituting the DEM results

1. DEM temperature

We will focus first on corrections to capture the DEM temperature. Because the behavior of the temperature is qualitatively different in the dense and dilute regimes, we will analyze each case separately. In the dilute regime, temperature is observed to drop upon increasing the friction
coefficient from zero, and following the approach of Jenkins and Zhang\textsuperscript{33} we seek to define an effective restitution coefficient that describes the total energy loss due to inelasticity and friction during an interparticle collision. To do so, we rewrite Eq. (15) by replacing $e$ in the $K$ term of the denominator with $e_{\text{eff}} = e_{\text{eff}}(e, \mu)$. The value of the effective restitution coefficient is then chosen to reproduce the temperature found in dilute-regime DEM simulations. Based on the DEM data, we find the expression

$$e_{\text{eff}} = e - f(\mu)$$

with

$$f(\mu) = \frac{3}{2} \mu \exp(-3\mu)$$

(23)

to provide a good fit. A comparison is made in Fig. 3 between this form and that derived by Jenkins and Zhang\textsuperscript{33} for small values of $\mu$. (We note that their expression for $f$ has a very small dependence on $e$, which we have neglected.) For $\mu \lesssim 0.2$, the two expressions show good agreement but begin to depart thereafter. In particular, following our data, our expression for $f(\mu)$ decreases for larger values of the friction coefficient, seeming to predict lower dissipation for more frictional particles. This nonmonotonic dependence is consistent with the previous finding that the translational temperature reaches the same steady-state values for the zero- and infinite-friction cases.\textsuperscript{34} We thus can write

$$T_{\text{dil DEM}} = \left( \frac{J}{K'} \right) (\dot{\gamma} d)^2$$

(24)

with

$$K'(\phi, e, \mu) = \frac{12}{\sqrt{\pi}} \phi^2 g_0 (1 - e_{\text{eff}}^2)$$

(25)

$$= K(\phi, e) \left( \frac{1 - e_{\text{eff}}^2}{1 - e^2} \right).$$

(26)

For volume fractions above $\phi \approx \phi_t$, an additional correction is needed to capture the increase in temperature with $\phi$ [Ref. 19], as the original GD model predicts a monotonically decreasing temperature. Plotting the dimensionless temperature versus the distance to the critical volume fraction $\phi_c - \phi$, as in Fig. 4, we observe that

$$T_{\text{dense DEM}} = \alpha_t (\dot{\gamma} d)^2 \left( \frac{1}{(\phi_c - \phi)^{1/2}} \right).$$

(27)
FIG. 4. Dense-regime temperature from DEM simulations. Symbols are the same as in Figs. 1 and 2. Scaled temperature versus volume fraction is shown for $\mu = 0.5$. A power law of slope $-1/2$ is observed (solid line) close to $\phi_c$, and the transition from dilute to dense behavior occurs for lower $\phi$ as $\epsilon$ decreases. The power-law fit is reasonably good for all cases of $\mu$.

A power law that is robust to changes in $\epsilon$ and $\mu$. The transition in temperature between the dilute and dense regimes can be captured conveniently by

$$T_{\text{DEM}} = \max(T_{\text{DEM}}^{\text{dil}}, T_{\text{DEM}}^{\text{dense}})$$  \hspace{1cm} (28)

or equivalently

$$T_{\text{DEM}} = M(\phi, \epsilon, \mu)(\dot{\gamma}d)^2$$  \hspace{1cm} (29)

with

$$M(\phi, \epsilon, \mu) = \max\left(\frac{J}{K^2}, \frac{\alpha_1}{\phi_c - \phi}^{1/2}\right).$$  \hspace{1cm} (30)

This form succeeds in capturing the sharp transition between the two behaviors as well as the loss of $\epsilon$-dependence observed in the close-packing limit, as seen in the comparison of Eq. (29) with the DEM temperature in Fig. 5. Moreover, it succeeds in reproducing the $\epsilon$-dependence of the volume fraction at which the dilute-to-dense crossover occurs without needing to specify this point explicitly. There is some minor quantitative disagreement in very dilute cases ($\phi \lesssim 0.2$), but we will neglect them here and reserve this topic for Sec. V.

FIG. 5. Comparison of the new steady-state temperature model with DEM results. Scaled temperature versus volume fraction is shown for (a) $\mu = 0.1$ and (b) $\mu = 0.5$. The agreement observed here between the KT and DEM predictions is substantially better than in Fig. 2. Symbols are the same as in Figs. 1 and 2.
examine the behavior of whether the effective restitution coefficient should appear here as well. To address this question, we are able to improve the kinetic-theory predictions of the pressure for the entire range of $\phi$ and for all $e$ and $\mu$ investigated (Fig. 8). As mentioned above for the temperature, there is some disagreement for $\phi \gtrsim 0.2$, though the $\mu$-independence of $p/\rho_s \phi T$ indicates that improved prediction of $T$ in this range would mitigate or eliminate this error.

Because the collisional contribution to the pressure contains a factor of $(1 + e)$, one could ask whether the effective restitution coefficient should appear here as well. To address this question, we examine the behavior of $p/\rho_s \phi T$ from dilute-regime DEM simulations as the friction coefficient is varied. According to the GD model $p/\rho_s \phi T = 1 + 2(1 + e)\phi g_0$, so replacement of $e$ here with $e_{\text{eff}}$ is
justified only if a \( \mu \)-dependence is observed. As seen in Fig. 6(b), no such dependence is observed in the simulations, and hence we maintain the original form.

3. DEM shear stress ratio

Finally, we must attempt to capture the DEM shear stress, which, since we have now described the pressure, can be achieved by simply modeling the shear stress ratio \( \eta \). Before investigating any further corrections, however, we first test whether the modifications done so far are sufficient to capture the DEM trends. In Fig. 9, we compare the DEM \( \eta \) values with the predictions of the original GD model (Eqs. (1) and (2)) augmented only with \( e_{\text{eff}} \) from Eq. (22), a chain length correction that reproduces the temperature as per Eq. (30), and the new \( g_0 \) expression from Eq. (31). This set of corrections would involve setting \( \delta_\tau = 1 \) and is analogous to the model of Jenkins and Berzi.\(^{31}\) As seen in the figure, there is somewhat of an improvement in that this modified GD model now predicts an increase in \( \eta \) due to friction in the dilute regime and a decrease in \( \eta \) as \( \phi \) approaches \( \phi_c \). However, this model continues to underpredict the effects of friction and still fails to produce a yield stress ratio at close packing. Hence, a further shear stress correction is necessary to reproduce the DEM \( \eta \) results quantitatively.
FIG. 9. Comparison of the DEM shear stress ratio values with the GD model modified to include only $e_{eff}$ and the new $g_0$ expression (i.e., $\delta_\tau = 1$ as per Jenkins and Berzi\textsuperscript{31}). The agreement observed here between the modified GD and DEM predictions is somewhat better than in Fig. 2, though the effect of friction is underestimated and a finite yield stress ratio is still not predicted. Results are shown for $\mu = 0.5$, though these trends are observed for all $\mu$. Symbols are the same as in Figs. 1 and 2.

The proposed $\eta$ model must not only predict the dilute- and dense-regime behaviors in the respective limits of $\phi \to 0$ and $\phi \to \phi_c$ but must also capture the transition between the two. In the case of pressure and temperature, the dilute-to-dense transitions involve changes of orders of magnitude, facilitating the bridging effort by allowing simple additive or switch models. However, the shear stress ratio varies over a much smaller range – always within the same order of magnitude – and the subtler transition will require a different bridging form.

In the dilute regime, we define $\psi(\mu, e, \phi) \equiv \eta(\mu, e, \phi)/\eta(\mu = 0, e, \phi)$ in order to quantify the departure of the shear stress ratio from the frictionless case. We then estimate a $\phi$-averaged value of $\psi$ for each case of $\mu$ and $e$, with weighting given more heavily to volume fractions between approximately between 0.2 and 0.4. The expression

$$\psi(\mu, e) = 1 + \frac{3}{10}(1 - e^2)^{-2/3}[1 - \exp(-8\mu)]$$

is found to provide a reasonably good fit while also satisfying some physical criteria, including the diminished influence of friction for more inelastic particles and the requirement that $\psi \to 1$ as $\mu \to 0$. The exponential $\mu$-dependence here is similar in form to that proposed for constitutive coefficients in a recent quasi-static model\textsuperscript{15}; it also agrees with our observations that the effect of $\mu$ on $\eta$ is monotonic, unlike for $T$, and saturates quickly as $\mu$ exceeds about 0.3. The dilute shear stress ratio is then written as

$$\eta_{\text{DEM}}^{\text{dilute}} = \eta_{\text{SS}}^{\text{GD}} \psi.$$  \hspace{1cm} (33)

In the dense regime, existing models for the shear stress ratio tend to take the form $\eta = \eta(I)$, where

$$I \equiv \frac{\dot{\gamma} \rho_d}{\sqrt{p/\rho_s}}$$

is the inertial number and can be interpreted as a ratio of the time scales of macroscopic shear deformation to microscopic particle rearrangement. Though this quantity has been shown to dictate granular rheology in the dense regime\textsuperscript{1,20-22} it is not particularly useful in characterizing dilute-regime behavior, in part because $I$ is not monotonic in $\phi$ over the entire range of volume fractions.\textsuperscript{45} Specifically, when evaluated using the steady-shear pressure, $I$ approaches zero in both the dense and dilute limits, achieving a maximum around $\phi \approx 0.2$. For this reason, we modify slightly the
definition of the inertial number to

\[ I' \equiv \frac{\dot{\gamma} d}{\sqrt{p/\rho_s}} = I/\phi \]  

(35)

in order to provide a monotonic \( I'(\phi) \) dependence. With \( I' \to 0 \) representing the dense limit and \( I' \to \infty \) representing the dilute limit, we now can model the transition from dense to dilute behavior using a function of \( I' \).

Conveniently, the original GD expression for the shear stress ratio can be expressed in terms of this inertial number. By combining Eqs. (1) and (2), we obtain

\[ \eta_{GD} = \beta_{GD}(\phi) I' \]  

(36)

with \( \beta_{GD}(\phi) \equiv \phi J/\sqrt{H} \). Similarly, our DEM \( \eta \) from Eq. (33) along with the DEM temperature from Eq. (29) yields

\[ \eta_{\text{dilute DEM}} = \beta(\phi) I' \]  

(37)

with

\[ \beta(\phi) \equiv \phi \psi J \sqrt{\frac{K}{K'H}} \]  

(38)

These expressions look similar in form to the dense-regime inertial-number model proposed by da Cruz et al.,\(^{21}\) which reads as

\[ \eta_{\text{da Cruz}} = \eta_s + \alpha I' \]  

(39)

except that the latter contains a constant yield stress ratio \( \eta_s \) added to the linear term. To obtain a comprehensive expression for \( \eta \), we therefore can proceed with a linear dependence on the inertial number and simply allow its prefactor to transition between dense and dilute expressions. We propose to achieve this transition using a blending function

\[ B(\alpha, \beta) = \alpha + (\beta - \alpha) \chi(I') \]  

(40)

defined in terms of a quantity \( \chi(I') \) that transitions from zero to unity according to

\[ \chi(I') = \frac{1}{(I_0/I')^{1.5} + 1} \]  

(41)

The forms of \( B \) and \( \chi \) are motivated by the \( \eta(I) \) models of Jop et al.\(^{22}\) and Chialvo et al.,\(^1\) though the 1.5-power from the latter is chosen because a power greater than unity is needed to achieve a complete \( \alpha \)-to-\( \beta \) transition. We then propose to write the shear stress ratio as

\[ \eta_{\text{DEM}} = \eta_s(\mu) \chi_s + B(\alpha, \beta) I' \]  

(42)

with

\[ \chi_s = 1 - \chi \]  

(43)

so that, as volume fraction decreases, the contributions of \( \eta_s \) and \( \alpha \) disappear commensurately. A comparison between this steady-state \( \eta \) model and the DEM results is shown in Fig. 10. As with the temperature and pressure, the fit for \( \eta \) is best for \( \phi \gtrsim 0.2 \). Additionally, though we find a small dependence of \( \alpha \) on \( \mu \) near \( \mu = 0 \), \( \alpha \) is nearly constant with respect to \( \mu \) for \( \mu \gtrsim 0.1 \); since most practical systems feature particles with friction coefficients in this range, we hence will assume \( \alpha \) to be constant as in Ref. 1.

D. Modified dynamic equations

Having now captured the steady-shear DEM results of \( T, p, \) and \( \eta \), we can now use these updated expressions to define correction factors to the dynamic GD equations. First, the pressure equation...
remains unchanged from Eq. (1) except that $g_0$ is given by Eq. (31). Next, we write the new shear stress expression as the sum of a yield stress and an inertial stress, i.e.,

$$ \tau = \tau_s + \tau_{\text{inertial}} $$

with

$$ \tau_s = \eta_s \chi_s \rho $$

and

$$ \tau_{\text{inertial}} = B(\alpha, \beta) I' \rho. $$

We also stipulate that only the inertial part $\tau_{\text{inertial}}$ will contribute to the viscous energy production rate— that is,

$$ J_{\text{vis}} = \tau_{\text{inertial}} \dot{\gamma}. $$

This decomposition, though not necessary to predict a minimum flow angle in KT-type models, is advantageous because it reproduces the observed dense-regime $\eta$ behavior while also leaving open the possibility for $\eta_s$ to be a function of microstructural variables (e.g., coordination number and fabric tensor) that become rheologically important in close-packed systems and that could evolve independently of the KT equations; it also maintains the linear term in $I'$ that naturally arises from the original GD model. As aforementioned, we can write $\tau_{\text{inertial}}$ alternatively as the GD expression for $\tau$ modified with a correction factor, i.e.,

$$ \tau_{\text{inertial}} = \tau_{\text{GD}} \delta_t $$

with $\tau_{\text{GD}}$ given by Eq. (2) and

$$ \delta_t = \frac{\alpha}{\beta} + \left[ 1 - \frac{\alpha}{\beta} \right] \chi. $$

Finally, the new energy dissipation rate expression is written as

$$ \Gamma = \frac{\rho_s}{d} K'(\phi, \epsilon, \mu) T^{3/2} \delta_{\Gamma}. $$

By solving the steady-state condition in Eq. (13) and requiring $T = T_{\text{DEM}}$ from Eq. (29) and $\tau/p = \eta_{\text{DEM}}$ from Eq. (42), we can easily define the correction factor

$$ \delta_{\Gamma} = \left( \frac{\beta \sqrt{H}}{K'M} \right) \delta_t. $$
We note that for frictionless particles in the dilute regime the original GD expressions are recovered. Additionally, the correction factor $\delta_{\lambda}$ is related to the chain length $L$ of Jenkins and Berzi, $^{31}$ with $L/d = \delta_{\lambda}^{-1}$. This chain length is equal to the particle diameter in the dilute limit and grows rapidly as $\phi \rightarrow \phi_c$. It also produces $T \sim g_0^{4/3}$ under steady shear, a result similar to Jenkins and Berzi’s $^{31}$ prediction that $T \sim g_0^{2/3}$.

It is important to mention that we have not addressed the need for corrections to the pseudo-thermal energy conductivity $\lambda$ to account for interparticle friction and dense packing. Because our simulation approach produces homogeneous shear with no temperature gradients, it cannot be used to measure conductivity. Hence, we are unable to assess the original model or recommend any new corrections. However, we briefly mention two simple options. First, Jenkins and Berzi $^{31}$ left the original GD expression for $\lambda$ unchanged, which is tantamount to setting $\lambda = \lambda_{GD} \delta_{\lambda}$ with $\delta_{\lambda} = 1$. The other option is to assume, based on the fact that the transport coefficients exhibit similar trends with respect to volume fraction, that $\delta_\lambda = \delta_\tau$. Though we cannot address this issue in the present study, we offer these two options and reserve their assessment for future work.

The complete model consists of the equations listed in Table I along with the constant parameters in Table II. As in the original GD model, all quantities are given as explicit functions of local system parameters (namely, volume fraction and shear rate) and particle-level properties (namely, material density, diameter, restitution coefficient, and friction coefficient).

### Table I. Summary of model equations.

<table>
<thead>
<tr>
<th>Primary equations</th>
<th>p = \rho_i HT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear stress:</td>
<td>$\tau = \tau_s + \tau_{\text{inertial}}$</td>
</tr>
<tr>
<td>$\tau_s = n_i \chi \mu$</td>
<td>$\tau_{\text{inertial}} = \beta I' \rho \delta_t$</td>
</tr>
<tr>
<td>Energy dissipation rate:</td>
<td>$\Gamma = \frac{\delta}{\Gamma_1} K' T^{3/2} \delta_c$</td>
</tr>
<tr>
<td>Viscous energy production rate:</td>
<td>$J_{vis} = \tau_{\text{inertial}} \gamma$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Secondary equations</th>
<th>$\delta_{\lambda} = \left(\frac{\rho_i}{\rho_i} \frac{\mu}{\mu} \right)^{1/2}$</th>
<th>$\delta_{\lambda} = \left(\frac{\rho_i}{\rho_i} \frac{\mu}{\mu} \right)^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{\lambda} = \left(\frac{\mu}{\mu} \frac{\rho_i}{\rho_i} \frac{\Gamma_1}{\Gamma_1} \right)^{1/2}$</td>
<td>$\delta_{\lambda} = \left(\frac{\mu}{\mu} \frac{\rho_i}{\rho_i} \frac{\Gamma_1}{\Gamma_1} \right)^{1/2}$</td>
<td></td>
</tr>
</tbody>
</table>

### Table II. Values of model constants. Values of the critical volume fraction $\phi_c$ and the yield stress ratio $\eta_s$ are taken from Ref. 1.

<table>
<thead>
<tr>
<th>$\mu$-dependent parameters</th>
<th>$\mu$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_c$</td>
<td>0.636</td>
<td>0.613</td>
<td>0.596</td>
<td>0.587</td>
<td>0.581</td>
<td></td>
</tr>
<tr>
<td>$\eta_s$</td>
<td>0.105</td>
<td>0.268</td>
<td>0.357</td>
<td>0.382</td>
<td>0.405</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$-independent parameters</th>
<th>$\alpha$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0$</td>
<td>0.2</td>
<td>0.36</td>
<td>0.06</td>
</tr>
</tbody>
</table>

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V. DISCUSSION

When applied to steady simple shear flows, the above dynamic model reproduces the temperature, pressure, and shear stress ratio results presented earlier in Figs. 5, 8, and 10 that the original GD theory is unable to predict. Moreover, the basic forms of the corrections used to achieve the improved performance all have precedent in the literature, as previous authors have proposed effective restitution coefficients to capture the effects of friction,\textsuperscript{31, 33} adjustable prefactors in the shear stress equation,\textsuperscript{7, 8, 44} a characteristic chain length in the energy dissipation equation,\textsuperscript{31} and expressions for the radial distribution function at contact that are based on simulation data.\textsuperscript{26, 29} Our work here builds upon these approaches and provides closures for the corrections in terms of local flow variables and particle properties.

An important finding of the present work is that corrections to the energy dissipation rate expression, such as those proposed by Jenkins and co-workers,\textsuperscript{31, 33} are insufficient on their own for extending traditional KT models to describe frictional and dense systems. Some aspects of the predictions are improved: the “chain length” correction reproduces an increase in temperature and a decrease in the shear stress ratio in the dense regime, and the effective restitution coefficient captures a decrease in temperature and pressure in the dilute regime as interparticle friction is increased. However, even when one constitutes these corrections based on DEM data, disparities between KT and DEM temperature and stresses remain. Hence, we choose to supplement the “chain length” and effective restitution coefficient with a modified radial distribution function at contact (to capture pressure), a correction factor to the shear stress, and decomposition of the shear stress into inertial and yield components. Though there are certainly other approaches one can take, they will necessarily involve modifications in the pressure and shear stress expressions. This point is consistent with the finding of Kumaran\textsuperscript{14, 25} that dense packing and particle roughness result in changes in the collision frequency and velocity distribution functions of a granular medium; in the full KT approach, the model expressions for energy dissipation rate, pressure, and shear stress are all derived from collision integrals involving the velocity distribution functions, so any change in the latter will necessarily produce changes in all of the former, not just the dissipation rate.

The proposed model has some key advantages over existing rheological models for granular materials. The first is its handling of the effects of friction. By acknowledging the decrease in $\phi_c$ upon increasing friction, the new model correctly predicts a corresponding increase in pressure in the dense regime. Previous kinetic theories accounting for friction neglect this effect and hence predict a decrease in pressure at all volume fractions resulting from increased dissipation and decreased temperature.\textsuperscript{31, 33} Second, while kinetic theory has long been used for modeling dilute flows, and while inertial-number models have proven successful for many dense flows, only very recently have the two been bridged to handle systems with wide variations in packing.\textsuperscript{46} Although this model did not account for interparticle friction or for dense-regime $e$-dependence, the good agreement between its predictions and experimental results for a number of flow problems is quite promising and motivates the more general bridging performed here. Finally, the recasting of the inertial-number description into a kinetic theory framework (1) makes the model compatible with boundary conditions involving granular temperature that are more sophisticated than the simple no-slip condition and (2) enables the model to account for nonlocal effects via the conduction of pseudothermal energy. Inertial-number models are typically coupled with the no-slip wall conditions whose validity may be questionable depending on system and particle properties (an issue we aim to address in a later work), while KT models afford the option of wall slip conditions connected to fluctuating energy and the inelasticity of particle-wall collisions.\textsuperscript{44, 47} This fluctuating energy, furthermore, can vary spatially and produce nonlocal rheological effects, which recent work has shown to be of substantial importance even in the dense systems for which inertial-number models were designed.\textsuperscript{23} While the “fluidity” approach in Ref. 23 (and also Ref. 48) is currently limited to the dense regime, the extended KT scheme presented here offers a way to account for nonlocality over the full range of volume fractions below jamming.

On the other hand, the proposed model does exhibit several weaknesses resulting in part from the approach taken in its development. Among the most salient of these is the overprediction of granular temperature in the very dilute regime ($\phi \lesssim 0.2$). Had we allowed the effective restitution...
coefficient to vary with volume fraction, we would have been able to capture the temperature more closely in this region. However, because the \( \phi \)-dependence is neither simple nor consistent across all cases of \( \epsilon \) and \( \mu \), we choose to define \( \epsilon_{\text{eff}}(\epsilon, \mu) \) as a constant, with higher importance placed on capturing cases where \( \phi \gtrsim 0.2 \). This definition is also more satisfying since \( \epsilon_{\text{eff}} \) is conceptually a particle property rather than an ensemble one. The disparity in very dilute cases, rather than arising from a poorly constituted \( \epsilon_{\text{eff}} \), is more likely the result of neglecting rotational degrees of freedom. The GD model is derived for smooth, frictionless particles and hence does not track rotational kinetic energy. In this context, conversion of translational energy to rotational energy is treated as an additional mode of translational energy dissipation, and so the parameter \( \epsilon_{\text{eff}} \) must encapsulate both real energy dissipation and loss to rotation. The extent of to which one mode predominates over the other may vary with volume fraction and therefore invalidate the assumption of a constant effective restitution coefficient. This problem is mitigated, though, by the fact that flow problems of practical interest manifesting such dilute regions tend to be heavily fluidized; hence, flow behavior in these regions is dictated primarily by the fluid-particle drag law, rendering any error in the solids-phase stress model rather inconsequential. Dense-phase predictions of the solids-phase stress, therefore, are most important to capture, and the corrections outlined in this work are weighted accordingly.

Another weakness of the present model is its reliance on empiricism, particularly in the corrections for the shear stress. Some of this need can likely be alleviated by considering rotational degrees of freedom, as the GD model and our modified version of it do not account for the dependence of the shear stress on the rotational temperature. It is possible, therefore, to start not with the GD model as the basis for modifications to account for friction and dense packing but rather with another existing KT model that considers particle rotation.\(^{10,11} \) The effects of friction can be included in such models by defining not only an effective normal restitution coefficient as done here but also an effective tangential restitution coefficient, which has been employed previously with some success.\(^{34} \) The inclusion of dense-regime effects can then be pursued as part of the present work by defining a “chain length” correction and a new \( g_0 \) expression. Even these tasks are encumbered by the empirical approach, as there is little agreement in the literature in regard to the precise dense-regime scalings for the stresses and temperature with respect to \( \phi_c - \phi \) [Ref. 49]. Our findings that \( T \sim (\phi_c - \phi)^{-1/2} \), \( g_0 \sim (\phi_c - \phi)^{-3/2} \), and \( \rho \sim (\phi_c - \phi)^{-2} \) in the dense regime are simply our best estimates based on our DEM data, and more work needs to be done to determine the reason for the variety of exponents proposed in the literature. Finally, the determination of \( \phi_c \) itself must be done empirically and as such has produced varying results. In particular, among recent works considering “rough”-particle shear flows, one finds that \( \phi_c \) is independent of the normal restitution coefficient and is purely a function of \( \mu \) [Ref. 1] while others show \( \phi_c \) to vary with both normal and tangential restitution coefficients.\(^{25,50} \) These works agree in their finding that roughness, whether measured by friction or tangential restitution, decrease \( \phi_c \) from its value in the smooth, frictionless case (\( \phi_c \approx 0.64 \)), but the contrary results regarding the influence of the normal restitution coefficient are hitherto unexplained and merit further study. The DEM data in the present work support the idea that \( \phi_c(\mu) \), and since these data form the basis for our empirical modifications to the GD model we move forward with such a form.

Related to the jamming point \( \phi_c \) is the rise of the particle stiffness \( k \) as a rheologically important parameter. The KT approach assumes that particle collisions are instantaneous and hence that particles are infinitely hard (i.e., \( k \rightarrow \infty \)). For this reason, in the present work we have made certain to consider only cases in which Bagnold scaling holds and hence the finite stiffness plays no role — that is, cases in the inertial regime. A truly complete rheological model, however, would be capable of describing both infinitely and finitely stiff particles and hence would span the inertial, quasistatic, and intermediate regimes. Much work has been done recently to elucidate differences between hard- and soft-particle rheology\(^{50,51} \) and to quantify the rheology of soft particles in elastic and inertial regimes.\(^{1,3,49,51-55} \) One recent work\(^{1} \) proposes a way to bridge the rheology of the inertial regime with the two elastic regimes by smoothly incorporating stiffness dependence to an extent governed by the dimensionless parameter \( \hat{\gamma} \equiv \gamma d/\sqrt{k} \rho_d \). Such an approach can be applied regardless of the choice of inertial-regime model, including the one described here, and could conceivably be coupled with more complex plasticity or hypoplasticity models for quasistatic flows.
A final drawback of the current approach is that the modifications made to the GD model, while appealing to micro- or mesoscopic concepts, are all rheological (i.e., macroscopic) in nature. In deriving KT models, one generally begins by describing the details of a single collision and the velocity distribution of a pair of particles — that is, model construction begins at the microscale and builds upwards. The approach taken here, though, makes measurements at the macroscopic level and uses them to constitute parameters that have a physical interpretation at smaller scales. The chain length correction, for example, can be interpreted as a mesoscopic length of a typical granular chain or microscopically as the result of dense-regime velocity correlations. The effective restitution coefficient is a microscopic metric of dissipation during the collision of two particles. The radial distribution function at contact $g_0$ is already microscopic in nature, and our modification to it accounts in a phenomenological way for anisotropy of particle collisions (which is also a microscale effect). The correction to the shear stress, on the other hand, involves a ratio of time scales of macroscopic shear and microscopic particle rearrangement. In some regard, however, the connection between our proposed values for the micro-/mesoscale quantities and their physically measurable counterparts is one of analogy more so than equality.

An alternative approach to the present one is, of course, to perform a rigorous KT derivation that includes provisions for friction and dense packing. The problem with this approach, as discussed in the Introduction, is the difficulty in obtaining a closed-form model. A collision integral involving a tangential impulse proportional to the normal one (via $\mu$) appears to require numerical solution, as does the integral for the non-Gaussian relative velocity distribution function that arises in dense flows — and even this distribution function is determined empirically. For the time being, the most promising approach toward developing a comprehensive rheological model appears to be modification of existing KT models for dilute flows of frictionless particles.

VI. SUMMARY

We investigate the influence of dense packing and interparticle friction on granular rheology in the inertial regime. From DEM simulations of homogeneous simple shear flows, we observe good agreement with the KT of GD in regard to predictions of pressure, shear stress ratio, and temperature in the case of frictionless particles below $\phi_f \approx 0.49$; however, for denser systems or for frictional particles, there is a substantial discrepancy between the simulation and GD results. Based on previous strategies in the literature for modifying the kinetic theory, we propose simple corrections to the GD model to bring its predictions closer to those of our DEM simulations. These corrections include effective restitution coefficient to capture the increased dissipation resulting from friction, a chain length correction to the energy dissipation rate, a new expression for the radial distribution function at contact, and a correction factor to the shear stress equation to reflect the influence of friction in the dilute regime as well as to reproduce inertial-number scalings in the dense regime. These new terms are then constituted in terms of particle-level properties and system parameters from our DEM simulations to produce a more complete rheological model. This model may then be implemented in continuum-model simulations to investigate large-scale flows of frictional particles over a wide range of volume fractions such as those found in numerous important flow problems including chutes, hoppers, and fluidized beds.

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