Limits of Computation
+
Course Recap

ORF 363/COS 323

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Reminder: NP-hard and NP-complete problems

Definition.

- A decision problem is said to be NP-hard if every problem in NP reduces to it via a polynomial-time reduction.
(roughly means “harder than all problems in NP.”)

Definition.

- A decision problem is said to be NP-complete if

(i)It is NP-hard

(ii)It is in NP.

(roughly means “the hardest problems in NP.”)

Remarks.

- NP-hardness is shown by a reduction from a problem that’s already known to be NP-hard.
- Membership in NP is shown by presenting an easily checkable certificate of the YES answer.
- NP-hard problems may not be in NP (or may not be known to be in NP as is often the case.)

The complexity class NP

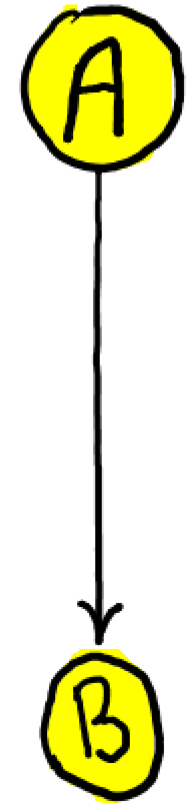
- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,…

$\in$P

- TSP
- MAXCUT
- STABLE SET
- SAT
- 3SAT
- PARTITION
- KNAPSACK
- IP
- COLORING
- VERTEXCOVER
- 3DMATCHING
- SUDOKU,…

NP-complete

Reductions



- A reduction from a decision problem A to a decision problem B is
    - **a "general recipe" (aka an algorithm)** for taking any instance of A and explicitly producing an instance of B, such that
    - the answer to the instance of A is YES if and only if the answer to the produced instance of B is YES.

- This enables us to answer A by answering B.

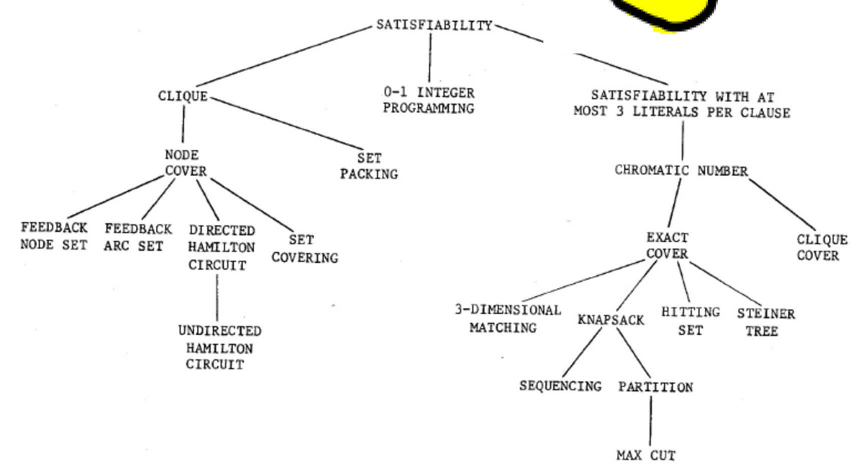


FIGURE 1 - Complete Problems

- Using reductions for showing NP-hardness:
    - If A is known to be hard, then B must also be hard.

P versus NP

- All NP-complete problems reduce to each other!
- If you solve one in polynomial time, you solve ALL in polynomial time!

- Assuming P≠NP, no NP-complete problem can be solved in polynomial time.
- This shows limits of *efficient* computation (under a complexity theoretic assumption)

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- **Today:** limits of computation in general (and under no assumptions)

Matrix mortality

Consider a collection of m $n \times n$ matrices $\{A_1, \dots, A_m\}$.

We say the collection is mortal if there is a finite product out of the matrices (possibly allowing repetition) that gives the zero matrix.

Example 1:

```
A1 =

     0     0
     0     1

A2 =

     0     1
    -1     0
```

```
>> A1*A2

ans =

     0     0
    -1     0

>> A1*A2*A1*A2

ans =

     0     0
     0     0
```

Example from [W11].

Mortal.

Matrix mortality

Consider a collection of m $n \times n$ matrices $\{A_1, \ldots, A_m\}$.

We say the collection is mortal if there is a finite product out of the matrices (possibly allowing repetition) that gives the zero matrix.

Example 2:

```
A1 =

     1    -2
     3     0

A2 =

     0    -1
    -1     0

A3 =

     1     2
     0    -1
```

Not mortal. (How to prove that?)

- In this case, can just observe that all three matrices have nonzero determinant.
- Determinant of product=product of determinants.

But what if we aren't so lucky?

```
>> A1*A2*A3

ans =

     2     5
     0     3

>> A1*A2*A3*A1*A3

ans =

    17    38
     9    18

>> A2*A2*A3*A1*A3

ans =

     7    16
    -3    -6

>> A2*A2*A1*A3

ans =

     1     4
     3     6

>> ...
```

Matrix mortality

- **MATRIX MORTALITY**

- **Input:** A set of m $n \times n$ matrices with integer entries.
- **Question:** Is there a finite product that equals zero?

Thm. MATRIX MORTALITY is **undecidable** already when

- $n = 3, m = 7,$

  or

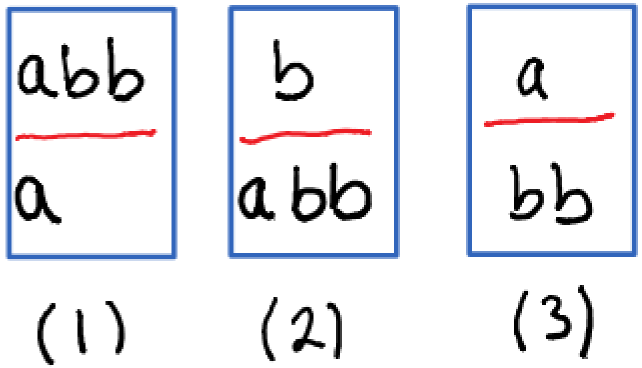
- $n = 21, m = 2.$

- This means that there is no finite time algorithm that can take as input two 21x21 matrices (or seven 3x3 matrices) and always give the correct yes/no answer to the question whether they are mortal.

- This is a definite statement.
  (It doesn't depend on complexity assumptions, like P vs. NP or anything like that.)

  - How in the world would someone prove something like this?
  - By a reduction from another undecidable problem!

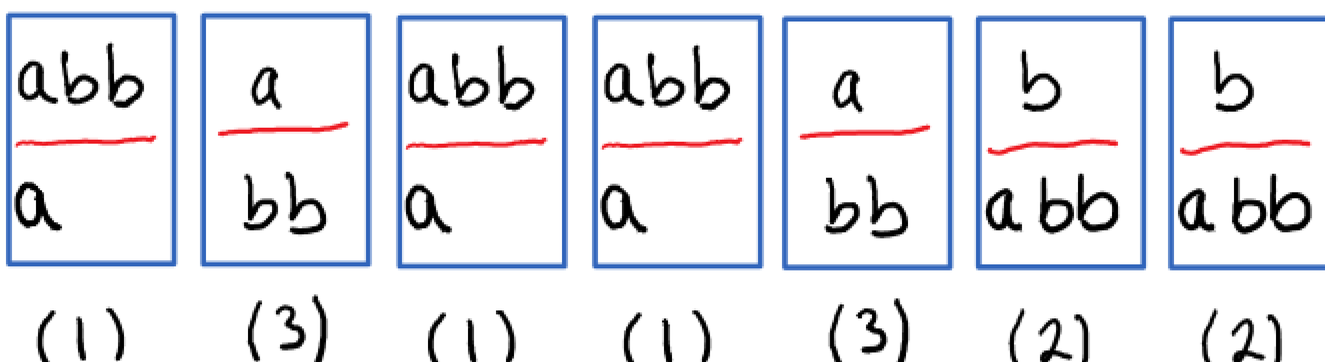
The Post Correspondence Problem (PCP)



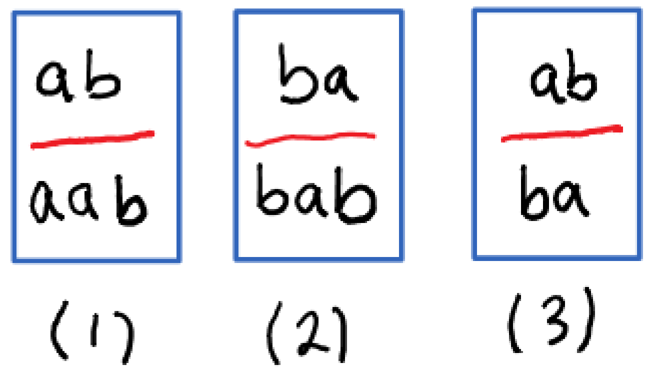
Emil Post
(1897-1954)

Given a set of dominos such as the ones above,
can you put them next to each other (repetitions allowed) in such a way that the top row reads the same as the bottom row?

Answer to this instance is YES:



The Post Correspondence Problem (PCP)



Emil Post
(1897-1954)

What about this instance?

Answer is NO. Why?

There is a length mismatch, unless we only use (3), which is not good enough.

But what if we aren't so lucky?

The Post Correspondence Problem (PCP)

- **PCP**

- **Input:** A finite set of m domino types with letters *a* and *b* written on them.

- **Question:** Can you put them next to each other (repetition allowed) to get the same word in the top and bottom row?

Emil Post
(1897-1954)

Thm. PCP is **undecidable** already when $m = 7$.

- Again, we are ruling out any finite time algorithm.

- PCP is decidable for $m = 2$.

- Status unknown for $2 < m < 7$.

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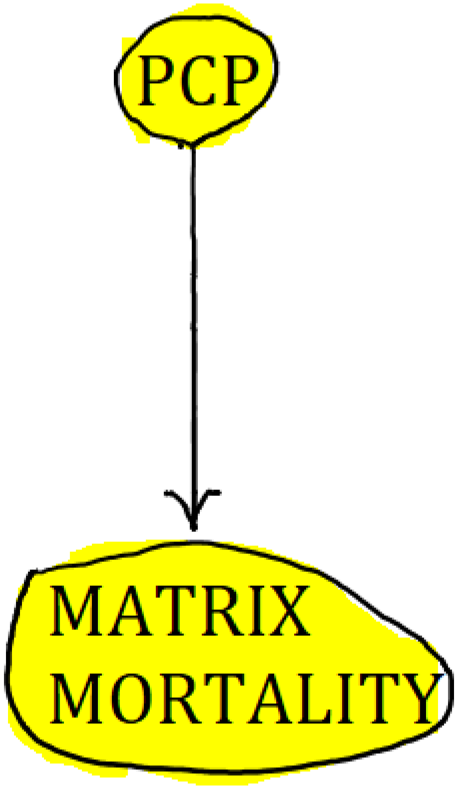
Reductions

- There is a rather simple reduction from PCP to MATRIX MORTALITY; see, e.g., [Wo11].



- This shows that if we could solve MATRIX MORTALITY in finite time, then we could solve PCP in finite time.
- It's impossible to solve PCP in finite time (because of another reduction!)
- Hence, it's impossible to solve MATRIX MORTALITY in finite time.
- Note that these reductions only need to be finite in length (not polynomial in length like before).

Integer roots of polynomial equations

- Can you give me three positive integers x, y, z such that

$$x^2 + y^2 = z^2?$$

- Sure: (3, 4, 5) (5, 12, 13) (8, 15, 17) (7, 24, 25) (20, 21, 29) (12, 35, 37) (9, 40, 41) (28, 45, 53)

And there are infinitely many more…

- How about $x^3 + y^3 = z^3$?
- How about $x^4 + y^4 = z^4$?
- How about $x^5 + y^5 = z^5$?

Fermat's last theorem tells us the answer is NO to all these instances.

Integer roots to polynomial equations

What about integer solutions to $x^3 + y^3 + z^3 = 29$?

YES: (3,1,1)

What about $x^3 + y^3 + z^3 = 30$?

Looped in MATLAB over all $|x, y, z|$ less than 10 million→ no solution!

But the answer is YES!! $(-283059965, -2218888517, 2220422932)$

What about $x^3 + y^3 + z^3 = 33$?

No one knows!

Integer roots of polynomial equations

▪**POLY INT**

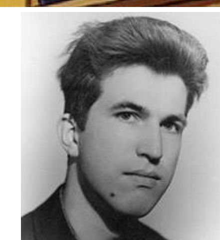
▪**Input:** A polynomial p in n variables and of degree d.

▪**Question:** Does it have an integer root?

- **Hilbert's 10th problem (1900):** Is there an algorithm for POLY INT?

From Logicomix

- **Matiyasevich (1970)** – building on earlier work by Davis, Putnam, and Robinson:
  No! The problem is undecidable.



- It's undecidable even in fixed degree and dimension (e.g., $d = 4, n = 58$).

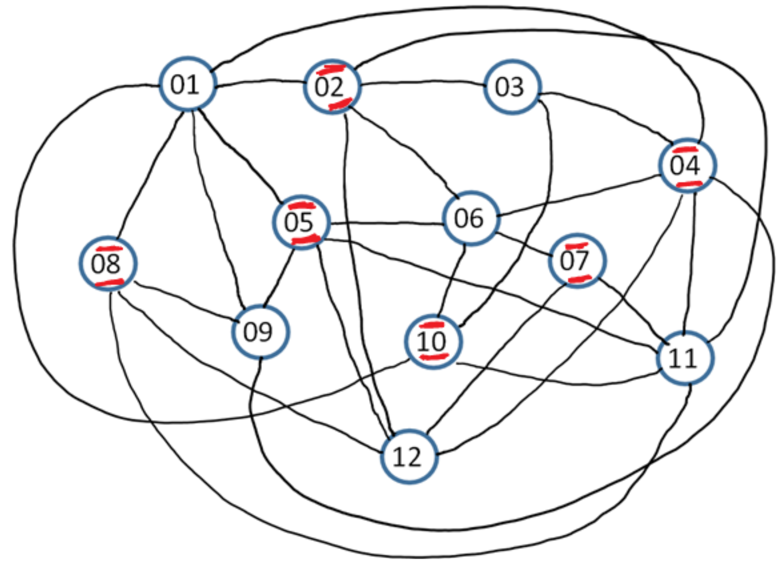
Real/rational roots of polynomial equations

- If instead of integer roots, we were testing existence of **real roots**, then the problem would become decidable.
  - Such finite-time algorithms were developed in the past century (Tarski–Seidenberg)
- If instead we were asking for existence of **rational roots**,
  - We currently don't know if it's decidable!

- Nevertheless, both problems are NP-hard. For example for
  - A set of equations of degree 2
  - A single equation of degree 4.
  - Proof on the next slide.

A simple reduction

- We give a simple reduction from STABLE SET to show that testing existence of a real (or rational or integer) solution to a set of quadratic equations is NP-hard.
- Contrast this to the case of linear equations which is in P.

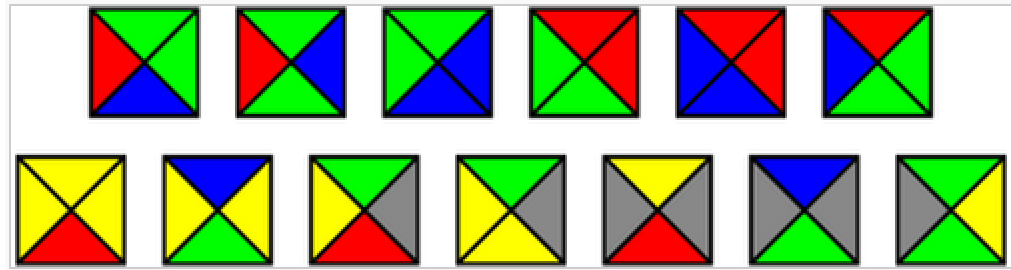


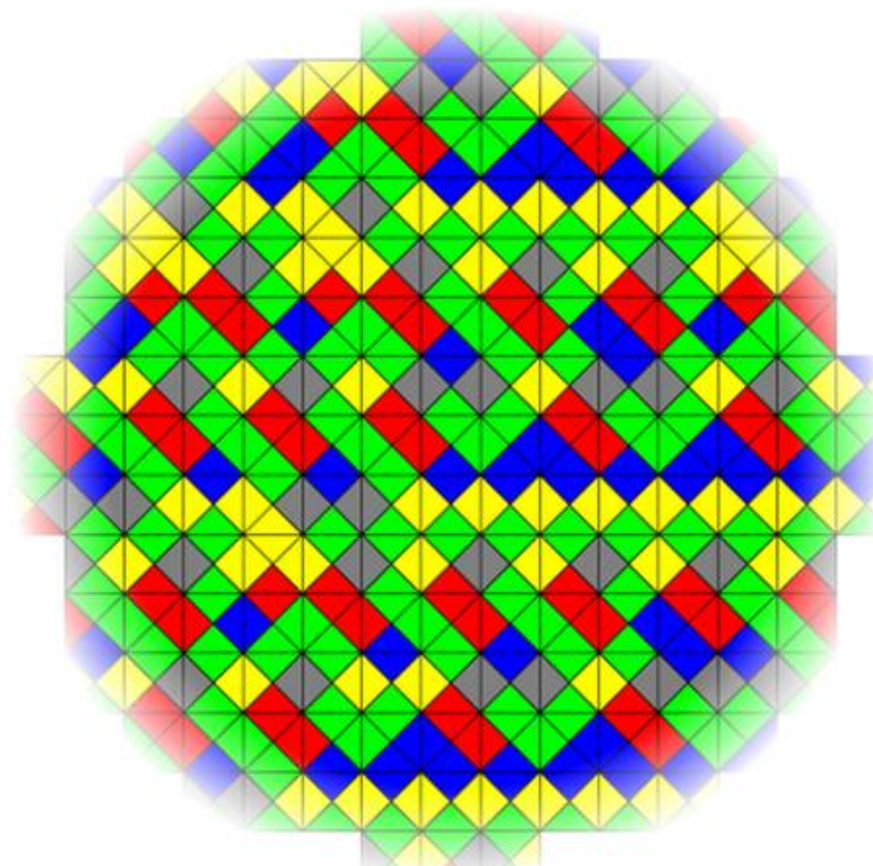
$$\exists \text{ stable set of size } k \iff \exists x \text{ s.t. } \left[\begin{array}{l} x_1 + \cdots + x_n = k \\ x_i + x_j \leq 1 \quad i,j \in E \\ x_i \in \{0,1\} \end{array}\right] \iff \exists x, z \text{ s.t. } \left[\begin{array}{l} (x_1 + \cdots + x_n - k)^2 = 0 \\ 1 - x_i - x_j = z_{ij}^2 \quad i,j \in E \\ x_i(1 - x_i) = 0 \quad i = 1, \ldots, n \end{array}\right]$$

- How would you go from here to a single equation of degree 4?

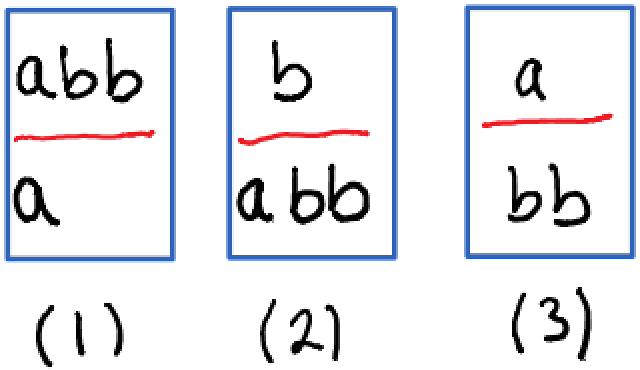
Tiling the plane

- Given a finite collection of tile types, can you tile the 2-dimenstional plane such that the colors on all tile borders match.
- Cannot rotate or flip the tiles.
- The answer is YES, for the instance presented.
- But in general, the problem is undecidable.



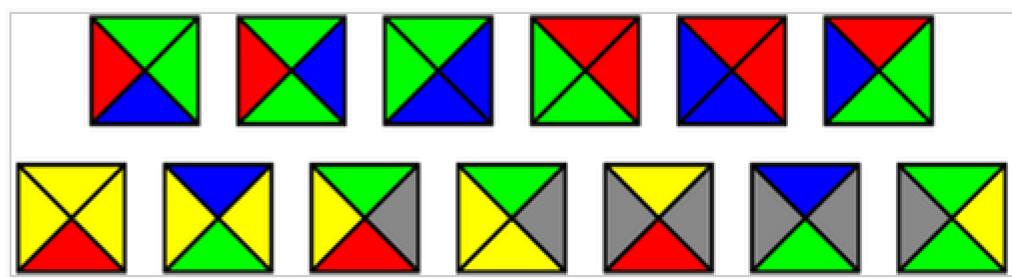


All undecidability results are proven via reductions



$$A1 = \begin{matrix} 1 & -2 \\ 3 & 0 \end{matrix} \qquad A2 = \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \qquad A3 = \begin{matrix} 1 & 2 \\ 0 & -1 \end{matrix}$$

$$x^3 + y^3 + z^3 = 33?$$



But what about the first undecidable problem?

The halting problem

- **HALTING**

- **Input:** A file containing a computer program p and a file containing an input x to the computer program.

- **Question:** Does p ever terminate (aka halt) when given input x?

An instance of HALTING:

```
function gradient_descent(x)

%gradient descent with exact line search for minimizing a quadratic
%function.
Q=[8 0;0 17];
b=[136;154];
xvec=[];
while norm(Q*x-b,2)>10^-5
    alpha=((Q*x-b)'*(Q*x-b))/((Q*x-b)'*Q*(Q*x-b));
    x=x-alpha*(Q*x-b);
    xvec=[xvec x];
end
```

Program p

x= [3;63];

The halting problem

An instance of HALTING:

```
function gradient_descent(x)

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     x=x-alpha*(Q*x-b);
     xvec=[xvec x];
end
```

Program p

x= [3;63];

- Both the program p and the input x can be represented with a finite number of bits.
- Can there be a program --- call it **terminates(p,x)** --- that takes p and x as input and always outputs the correct yes/no answer to the question: does p halt on x?

  - We'll show that the answer is no!
  - This will be a proof by contradiction.

The halting problem is undecidable

Proof.

- Suppose there was such a program terminates(p,x).
- We'll use it to create a new program paradox(z):

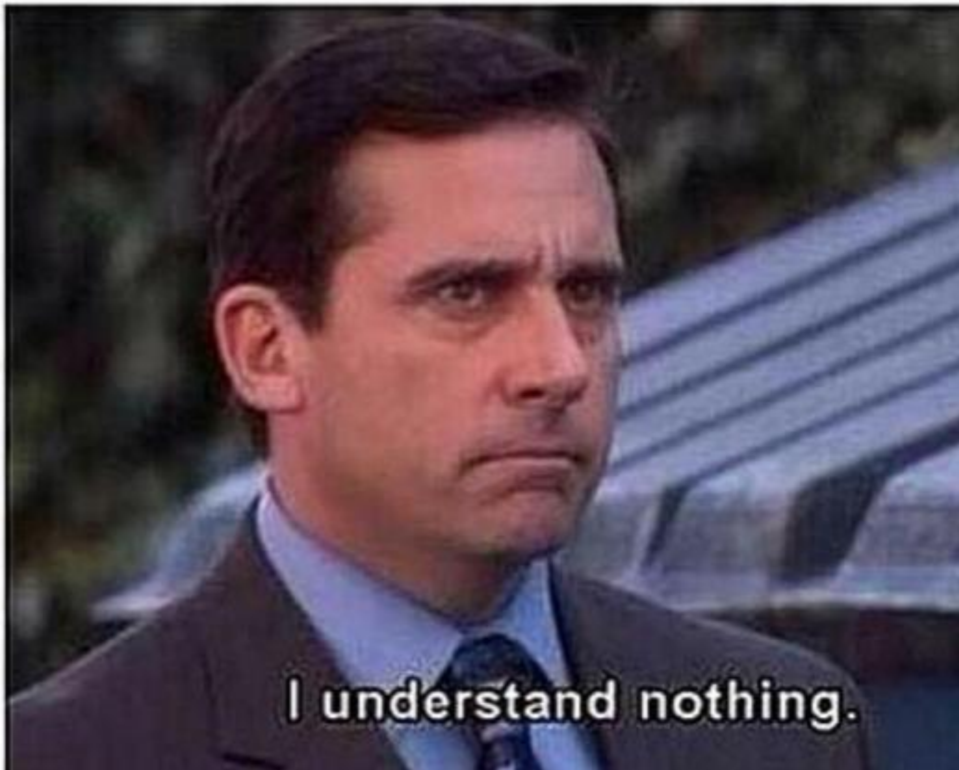
```
function paradox(z)

1: if terminates(z,z)==1 goto line 1.
```

- The input z to paradox is a computer program.
- As a subroutine, paradox asks terminates to check whether a given computer program z halts when given itself as input. (This is perfectly legal as any program is just a finite number of bits.)
- Note that paradox halts on z if and only if z does not halt when given itself as input.

  - What happens if we run paradox(paradox) ?!
    - If paradox halts on itself, then paradox doesn't halt on itself.
    - If paradox doesn't halt on itself, then paradox halts on itself.
    - This is a contradiction→ terminates can't exist.

Typical 1st time reaction to the proof of the halting problem



The halting problem (1936)

Alan Turing
(1912-1954)

A simpler story to tell strangers at a bar…

(aka Russell’s paradox)

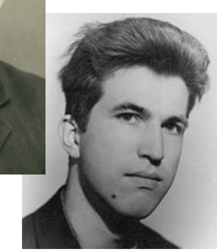
The power of reductions (one last time)

A simple paradox/puzzle: 

```
function paradox(z)
1: if terminates(z,z)==1 goto line 1.
```

(lots of nontrivial mathematics,

including the formalization of the notion of an "algorithm")



A fundamental algorithmic question:

- **POLY INT**
- **Input:** A polynomial p in n variables and degree d.
- **Question:** Does it have an integer root?

A remarkable implication of this…

Take your favorite long-standing open problem in mathematics:
e.g.,

- Is there an odd perfect number? (an odd number whose proper divisors add up to itself?)
- Is every even integer >2 the sum of two primes? (the Goldbach conjecture)

In each case, you can explicitly write down a polynomial of degree 4 in 58 variables, such that if you could decide whether your polynomial has an integer root, you would have solved the open problem.

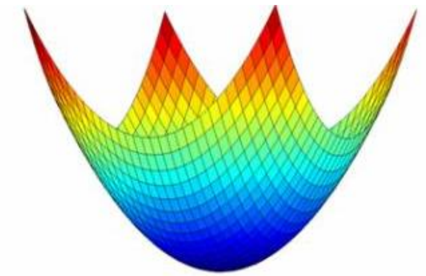
Proof.

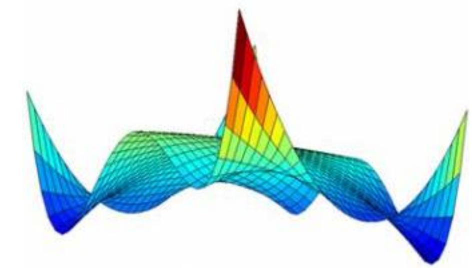
1) Write a code that looks for a counterexample.
2) Code does not halt if and only if the conjecture is true (one instance of the halting problem!)
3) Use the reduction to turn into an instance of POLY INT.

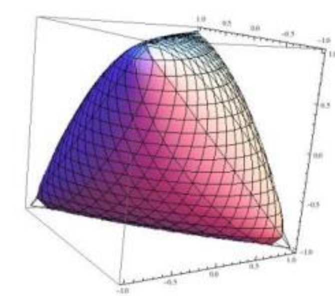
A look back at ORF 363/COS 323

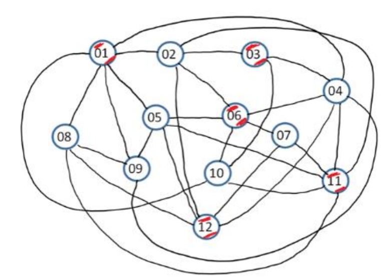
Topics we covered in optimization

- Optimality conditions for unconstrained optimization
- Convex analysis
  - Convex sets and functions
  - Optimality conditions for constrained convex problems
  - Convexity detection and convexity-preserving rules
- Modeling a problem as a convex program
  - Solving it in CVX or CVXPY
- Algorithms for convex unconstrained optimization
- Algorithms for constrained linear optimization
- Semidefinite programming
- Convex relaxations for non-convex and combinatorial optimization
- Theory of NP-completeness
- Undecidability



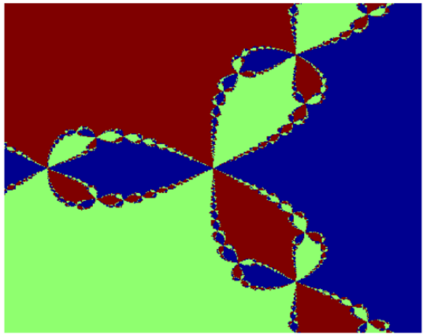


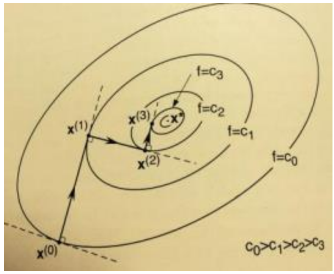


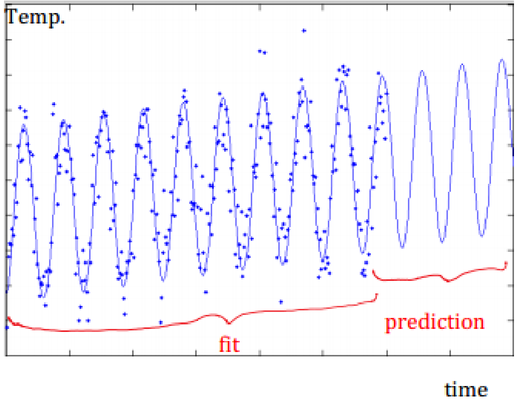


Topics we covered in numerical computing

- Least squares
  - Optimality conditions and normal equations
- Singular value decomposition
- Solving linear systems
- Iterative descent methods
- Root finding
  - Bisection, the secant method
  - The Newton method, Newton fractals
- Nonlinear least squares
  - The Gauss-Newton method
- Convergence analysis
  - Convergence rates of gradient descent and Newton
  - Condition number
- Approximation and fitting







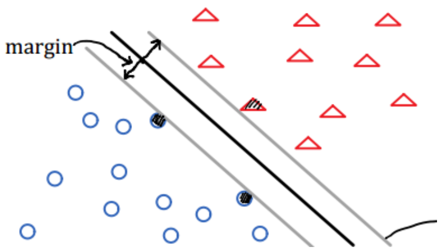
Applications of these tools are ubiquitous…

Hey man,

I'm tired of this homework for ORF 363. Let's go party tonight. We can always ask for an extension.

-J

Spam

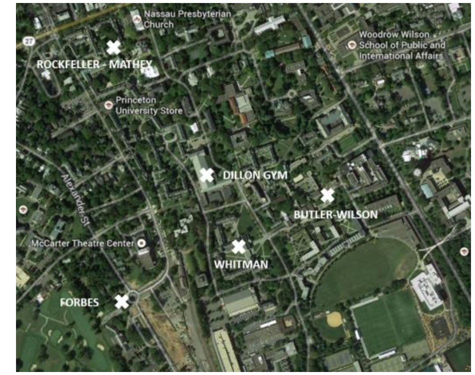


Support vector machines

k=130

Original

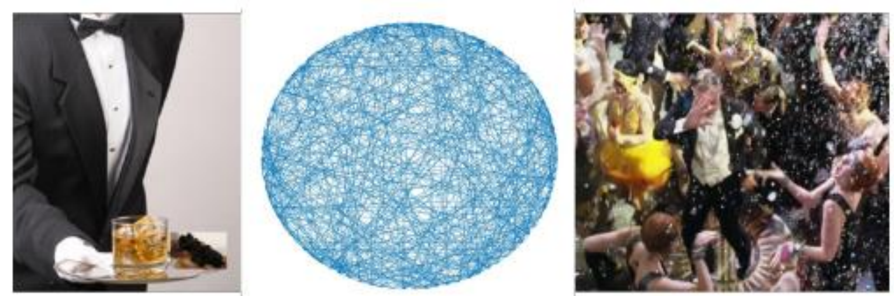
Image compression



Optimal facility location

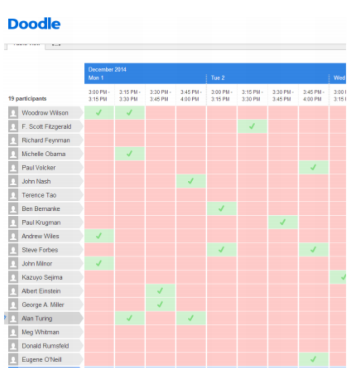
Hillary vs. Bernie

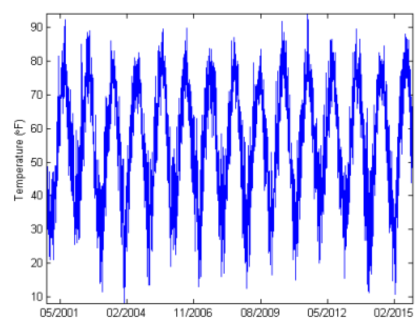
Fairness in grading



Event planning

Scheduling





Optimal control

Minimum intensity radiation therapy

Bookmaking

Portfolio optimization

…

We met lots of mathematicians!

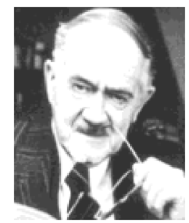

















Who is who?

And on what topic did they feature in this class?

How to check if an optimization problem is easy?

- Check if it’s convex!
- The functional form of convexity meant:
  - Objective a convex function (if you are minimizing)
  - Constraints: “Convex≤Concave”, “Affine==0”.
- If it is, then (most of the time) CVX can already solve it for you up to a reasonably large size.
- There are occasional exceptions:
- Nonconvex problems can be easy:
  - Singular value decomposition (best rank r approximation to a given matrix)
  - One can argue that there is “hidden convexity” (e.g., the dual is an SDP)
- Convex problems can be hard:
  - Optimizing over the set of nonnegative polynomials or copositive matrices
  - Not quite in functional form, but they can be made as such.
- Checking convexity may not be easy
- But the calculus of convex functions and convexity-preserving rules often suffice.

How to check if an optimization problem is easy (formally)?

- Can you reduce it to a problem in P?
- If yes, then it's often easy
  - Unless the polynomial in the running time has high degree or large constants—often rare

- Can you show it's NP-hard?
- You must reduce a different NP-hard problem to it.
  - If you succeed, an exact efficient algorithm is out of the picture (unless P=NP)

- NP-hard problems still routinely solved in practice.
- Workarounds: heuristics, solving special cased exactly, convex relaxations.

- Convex optimization is often a powerful tool for approximating non-convex and NP-hard problems.
- We saw many examples in recent weeks; e.g., LP and SDP relaxations.

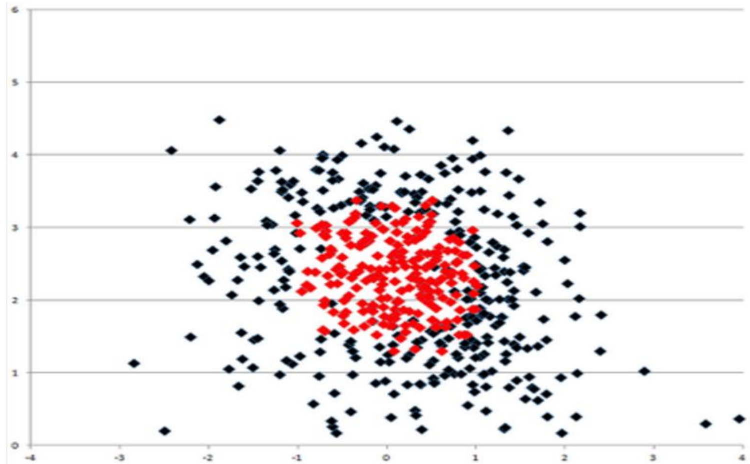
Slide from lecture 1: Course objectives

- The skills I hope you acquire:

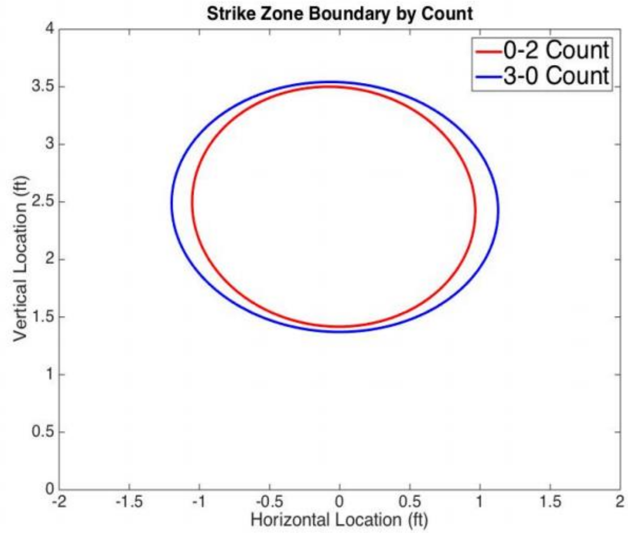
- Ability to view your own field through the lens of optimization and computation
    - To help you, we'll draw applications from operations research, statistics, economics, machine learning, engineering, …
- Learn about several topics in scientific computing
- More mathematical maturity and ability for rigorous reasoning
    - There will be some proofs in lecture. Easier ones on homework.
- Enhance your coding abilities
    - There will be a coding component on every homework and on the take-home final.
- Ability to recognize hard and easy optimization problems
- Ability to use optimization software
    - Understand the algorithms behind the software for some easier subclass of problems.

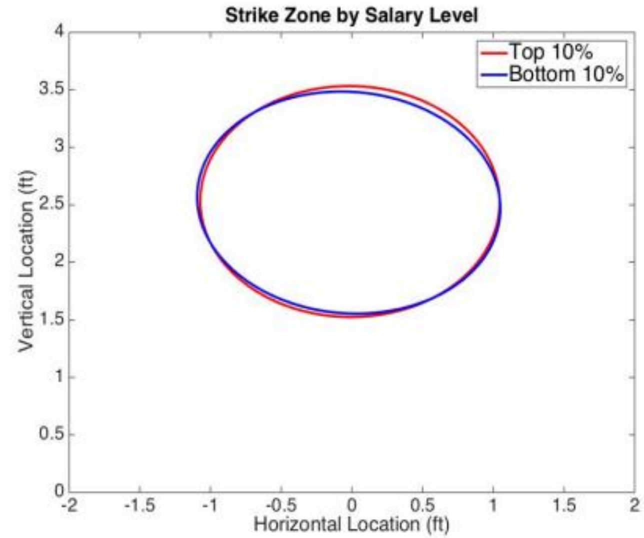
An example: Jacob Eisenberg's work

- The "real strike zone" in major league baseball!



Robust minimum-volume ellipsoids obtained from *semidefinite programming*





The final exam!

- Take-home. No collaboration allowed. Can only ask clarification questions as public questions on Ed Discussion. Can use all lecture notes, psets/previous exam solutions, and reference books of the course. Can only use "Google/ChatGPT" for problems with MATLAB/Python/software (although even that should not be needed).
- Exam will go out on **Saturday, December 14, 8AM EST.**
- Have to take it in **48 consecutive hours** (clock starts when you download).
- To be submitted on Gradescope as a single PDF file.
  - Keep an electronic copy of your exam.
- **Latest submission time is Thursday, December 19, 11:59PM EST** (University deadline).
- Don't forget that pset 8 is due Wednesday, December 11, at 1:30PM EST.

What to study for the final?

- All the lecture notes.
- Psets 1-8, practice exams (we have posted several!).
- If you need extra reading, the last page of the notes points you to certain sections of the book for additional reading.
- Be comfortable with MATLAB/Python and CVX/CVXPY. Make sure your software is running.

Some good news

- Undecidability from today's lecture won't be on the final.
- Theory of NP-completeness won't be on the final (but it is on HW 8).
- Lecture 10 (conjugate gradients), Lecture 12 (duality), and the "additional slides on applications of sum of squares optimization" are optional and not on the final.
- Six practice final exams (with solutions) are already posted.
- The TAs and I will hold office hours throughout reading period and up to the day of the day of the exam. Regular schedule (see syllabus, or slides of lecture 1).

- In addition, we will have the following review sessions:
  Burak (pset 1&2) Friday Dec 6, 1:30-3:30 PM EST, Friend 008
  George (pset 3&4) Monday Dec 9, 1:30-3:30 PM EST, Friend 008
  Ben (pset 5&6) Tuesday Dec 10, 1:30-3:30 PM EST, Friend 008
  Ben (psets 7&8) Wednesday Dec 11, 1:30-3:30 PM EST, Friend 008
  Yixuan (past finals) Thursday Dec 12, 1:30-4:30 PM EST, Friend 008
  AAA **(comprehensive review)** Friday Dec 13, 6-9 PM EST, Friend 008

There will be pizza!

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Last but not least…

- It was great having you all in my class.
- Thank you for making this an enjoyable and rewarding semester!

- Go make optimal decisions in your lives! (Make sure you optimize for the right objective functions!)
- And keep in touch!

AAA.
December 5, 2024

Notes & References

▪Notes:

- Chapter 8 of [DPV08] mentions undecidability and the halting problem. Chapter 9 of [DPV08] is optional but a fun read.

▪References:

-[Wo11] M.M. Wolf. Lecture notes on undecidability, 2011.

-[Po08] B. Poonen. Undecidability in number theory, *Notices of the American Mathematical Society,* 2008.

-[DPV08] S. Dasgupta, C. Papadimitriou, and U. Vazirani. Algorithms. McGraw Hill, 2008.