Sum of Squares Optimization and Its Applications

Amir Ali Ahmadi

Princeton University

Dept. of Operations Research and Financial Engineering (ORFE)

ORF 363

Optimization over nonnegative polynomials

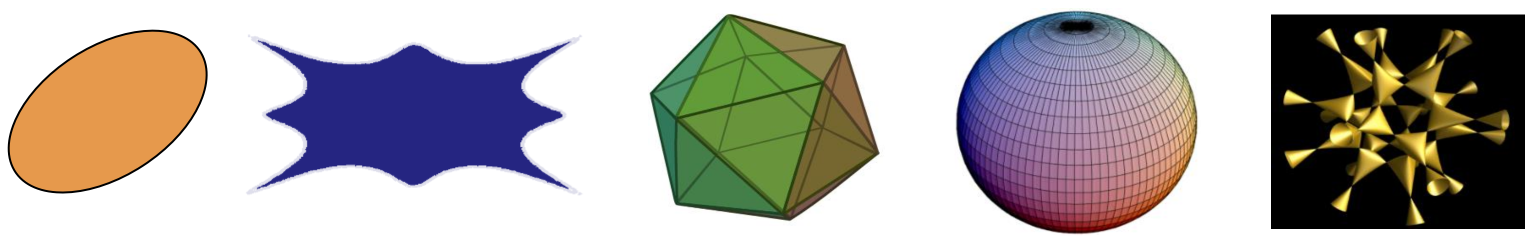
Definition by example: How to pick c_1, c_2, c_3 so to make

$$p(x_1, x_2) = c_1 x_1^4 - 6x_1^3 x_2 - 4x_1^3 + c_2 x_1^2 x_2^2 + 10x_1^2 + 12x_1 x_2^2 + c_3 x_2^4$$

nonnegative over a given basic semialgebraic set?

Basic semialgebraic set: $\{x \in \mathbb{R}^n |\ g_i(x) \geq 0, h_j(x) = 0\}$

Ex: $x_1^3 - 2x_1 x_2^4 \geq 0$
$x_1^4 + 3x_1 x_2 - x_2^6 \geq 0$



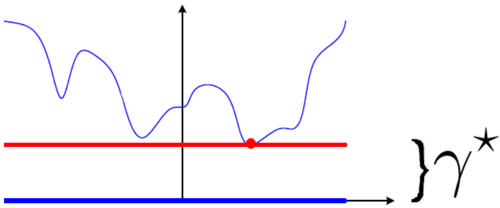
-This problem is fundamental to many areas of applied/computational mathematics.
-It is the problem that "SOS optimization" is designed to solve.

Why would you want to do this?!

- Let's start with five application domains…

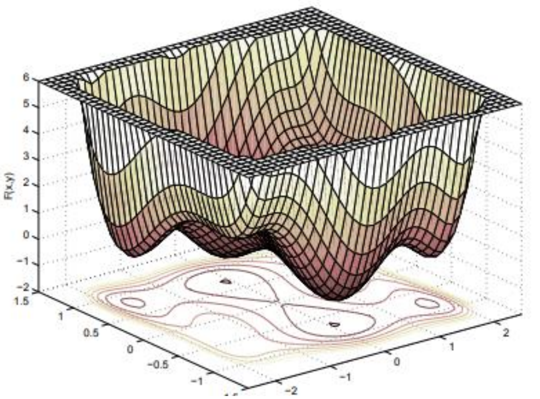
1. Polynomial optimization

$$\left[\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & p(x) \\ \text{s.t.} & g_i(x) \geq 0 \\ & h_j(x) = 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{ll} \max\limits_{\gamma \in \mathbb{R}} & \gamma \\ \text{s.t.} & p(x) - \gamma \geq 0 \quad \forall x \in K \end{array}\right]$$



$p, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ polynomials

Feasible set $K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_j(x) = 0\}$



▪**Many applications:** finding equilibria in games, the optimal power flow problem, low-rank matrix factorization, dictionary learning, training of deep nets with polynomial activation function, sparse regression with nonconvex regularizes, etc.

▪Intractable in general (includes all NP-complete problem)

2. Optimization under input uncertainty

How to make optimal decisions when input to optimization problem is uncertain/noisy?

Example: The Markowitz portfolio optimization problem

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n,\ \gamma \in \mathbb{R}} \quad & \gamma \\
\text{s.t.} \quad & \mu^T x \geq \gamma \quad \text{(return)} \\
& x^T \Sigma x \leq \delta \quad \text{(risk)} \\
& x \geq 0,\ \sum_{i=1}^n x_i = 1 \\
& x \in \Omega
\end{aligned}
$$

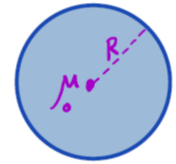
$\mu \in \mathbb{R}^n$: mean vector of the returns

$\Sigma \in \mathbb{S}^{n \times n}$: covariance matrix of the returns

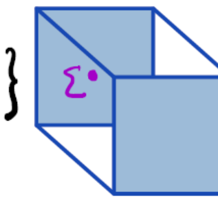
In practice estimated from past data/ML model. Optimal portfolio sensitive to this input.

Accounting for uncertainty:

$$U_\mu = \{ \mu_0 + u \in \mathbb{R}^n \mid \|u\| \leq R \}$$



$$U_\Sigma = \{ \Sigma \in S^{n \times n} \mid \Sigma \succeq 0,\ \Sigma_{ij}^l \leq \Sigma_{ij} \leq \Sigma_{ij}^u \}$$

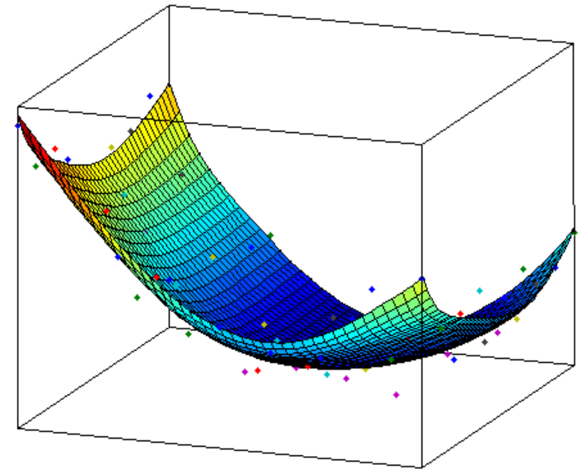


$$
\begin{aligned}
\max_{x \in \mathbb{R}^n,\ \gamma \in \mathbb{R}} \quad & \gamma \\
\text{s.t.} \quad & \mu^T x \geq \gamma \quad \forall \mu \in U_\mu \\
& x^T \Sigma x \leq \delta \quad \forall \Sigma \in U_\Sigma \\
& x \geq 0,\ \sum_{i=1}^n x_i = 1 \\
& x \in \Omega
\end{aligned}
$$

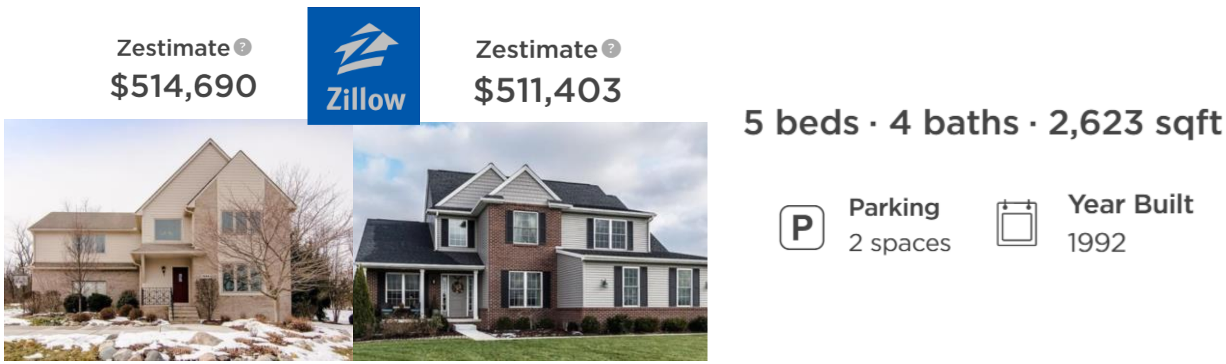
3. Statistics and machine learning

Shape-constrained regression; e.g., *monotone and/or convex regression*

Shape constraints act as regularizer, improve test performance, make model more interpretable and trustworthy



Example 1: Shape constraints natural in most applications



Monotonicity of a polynomial $p(x_1, \ldots, x_n)$ with respect to feature j: $\dfrac{\partial p(x)}{\partial x_j} \geq 0, \forall x \in B$

Example 2: "ML for fast real-time convex optimization"

$$g(b) := \min_{x \in \mathbb{R}^n} f_0(x)$$
$$\text{s.t. } f_i(x) \leq b_i \; i = 1, \ldots, m$$
$$x \in \Omega$$

$f_0, \ldots, f_m$ convex functions, Ω a convex set.

Goal: learn $g(b)$ offline from training set; evaluate it online very fast

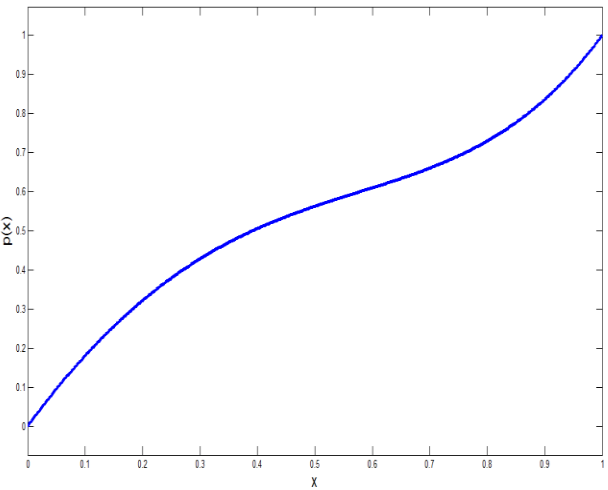
$g: \mathbb{R}^m \to \mathbb{R}$ is
- convex
- nonincreasing w.r.t. all arguments

$y^T \nabla^2 g(b) y \geq 0, \forall b, \forall y$ $\quad \dfrac{\partial g(b)}{\partial b_j} \leq 0, \forall b, \forall j$

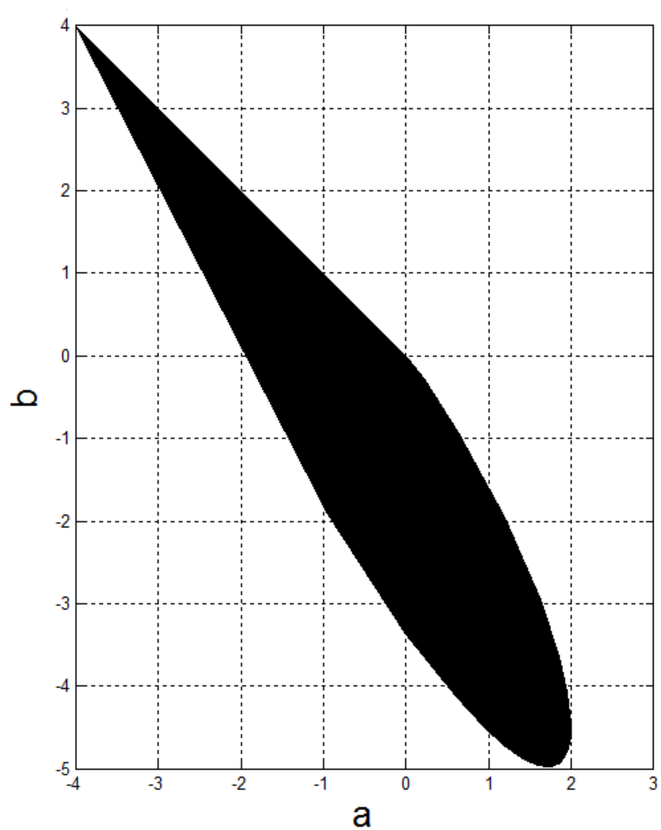
Imposing monotonicity

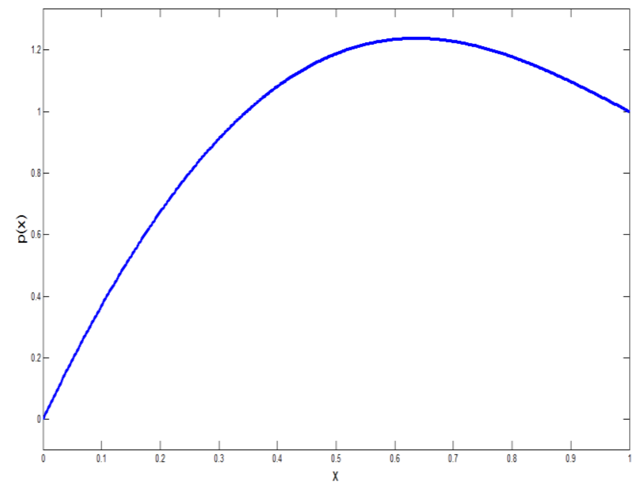
- For what values of a, b is the following polynomial monotone over $[0,1]$?

$$p(x) = x^4 + ax^3 + bx^2 - (a+b)x$$



$a = 0, b = -2$

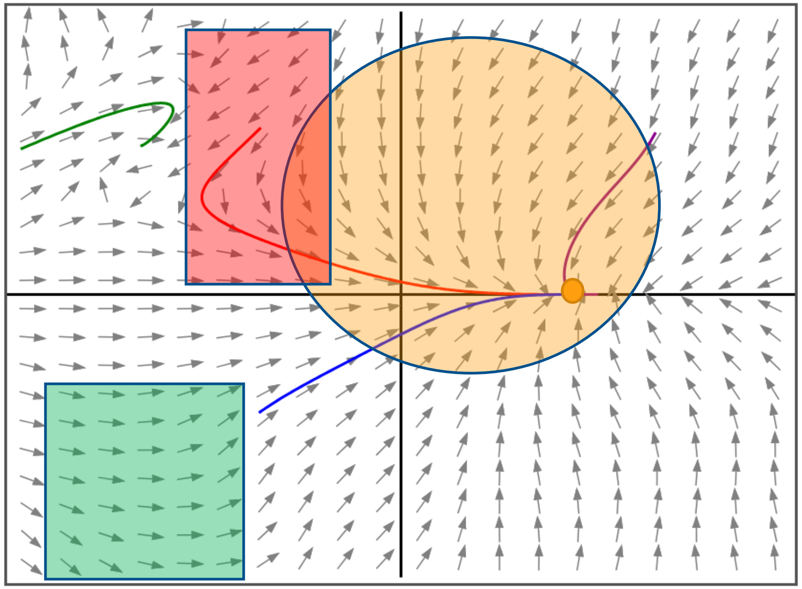




$a = -1, b = -3$

4. Certifying properties of dynamical systems

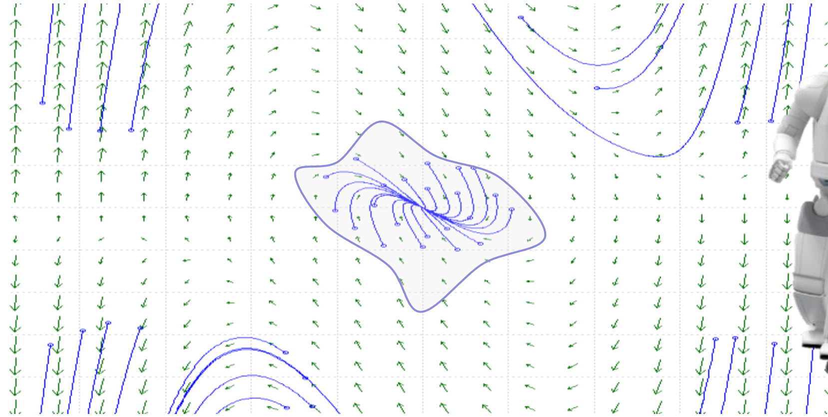
$$\dot{x} = f(x)$$

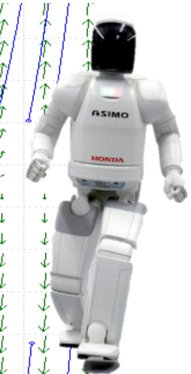


Questions about properties of dynamical systems (e.g., stability, safety) → *Lyapunov theory* → Search for functions satisfying nonnegativity constraints → *Semialgebraic parametrization* → Polynomial inequalities

Example: certifying stability

$$\dot{x} = f(x) \qquad f : \mathbb{R}^n \to \mathbb{R}^n$$



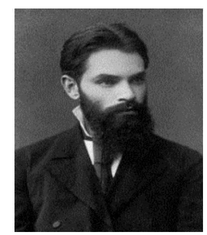


Ex. $\dot{x}_1 = -x_2 + \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 x_2$

$\dot{x}_2 = 3x_1 - x_1 x_2$

Locally asymptotic stability (LAS) of equilibrium points

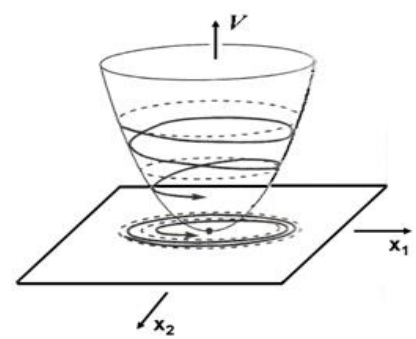
Lyapunov's theorem (and its converse):



The origin is LAS if and only if there exists a C^1 function $V: \mathbb{R}^n \to \mathbb{R}$ that vanishes at the origin and a scalar $\beta > 0$ such that

$$V(x) > 0$$

$$V(x) \leq \beta \Rightarrow \dot{V}(x) = \nabla V(x)^T f(x) < 0$$



(If $\dot{V}(x) < 0$ everywhere, then globally stable.)

Example: certifying collision avoidance

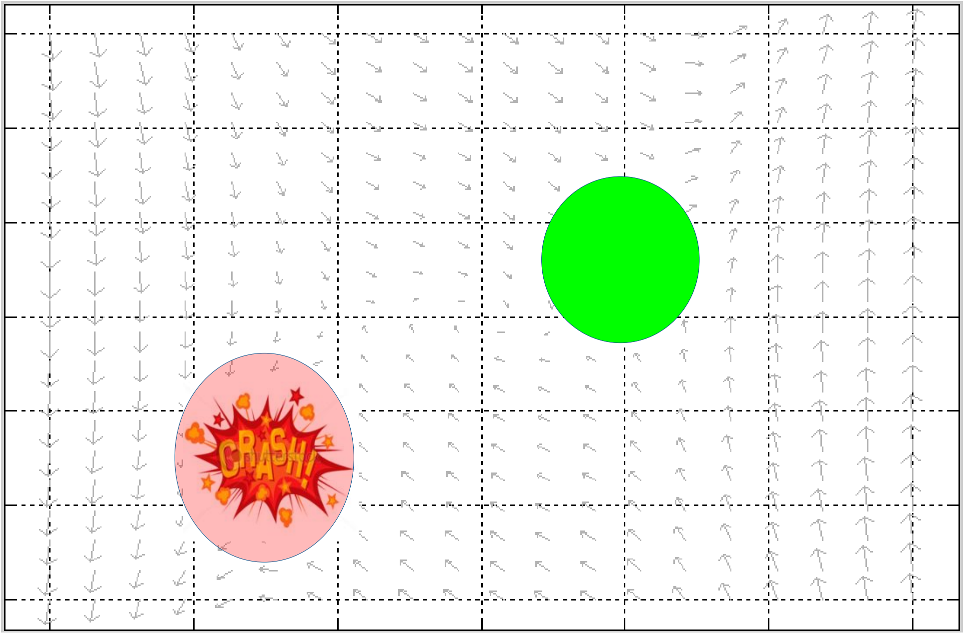
$$\dot{x} = f(x)$$

(vector valued polynomial)

$\mathcal{S}$: needs safety verification

$\mathcal{U}$: unsafe (or forbidden) set

(both sets basic semialgebraic)



Safety guaranteed if we find a "Lyapunov function" such that:

$$B(\mathcal{S}) < 0$$

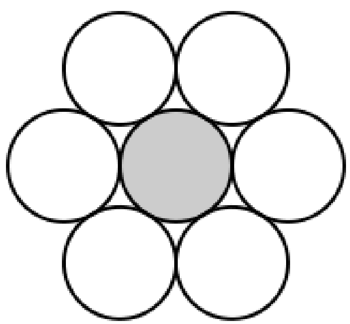
$$B(\mathcal{U}) > 0$$

$$\dot{B} = \langle \nabla B(x), f(x) \rangle \leq 0$$

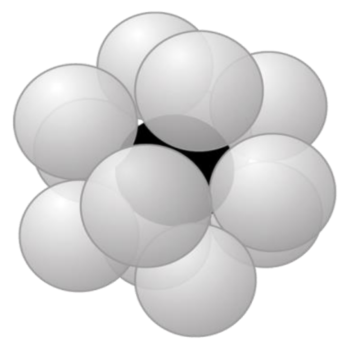
5. Automated theorem proving in geometry

- **Kissing number in dimension n:** largest number of n-dimensional non-overlapping spheres that can simultaneously touch (or "kiss") a common unit sphere.

$k_2 = 6$



$k_3 = ?$



$k_3 \geq 12$

Newton

$k_3 = 12$

Gregory

$k_3 = 13$

Discussion/bet in 1694

Newton proved to be correct in 1953!

13 spheres impossible iff the following system is ***infeasible***:

$$x_i^2 + y_i^2 + z_i^2 = 4, \; i = 1, \ldots, 13$$
$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \geq 4,$$
$$i, j \in \{1, \ldots, 13\}^2$$

$$\boxed{\begin{array}{c}\begin{cases} g_1(x) \geq 0 \\ \vdots \\ g_{100}(x) \geq 0 \end{cases} \\ \text{infeasible}\end{array}} \Leftrightarrow \boxed{\begin{bmatrix} g_1(x) \geq 0 \\ \vdots \\ g_{99}(x) \geq 0 \end{bmatrix} \Rightarrow g_{100}(x) < 0}$$

Outline

- **Global nonnegativity**
  - Sum of squares (SOS) and semidefinite programming
  - Two applications
  - Hilbert's 17^{th} problem
- **Nonnegativity over a region**
  - Putinar's Positivstellensat
  - Two applications
- **Recap and further reading**

How would you prove nonnegativity?

Ex. Decide if the following polynomial is nonnegative:

$$\begin{aligned} p(x) &= x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ &\quad -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \end{aligned}$$

- Not so easy! (In fact, **NP-hard for degree ≥ 4**)
- But what if I told you:

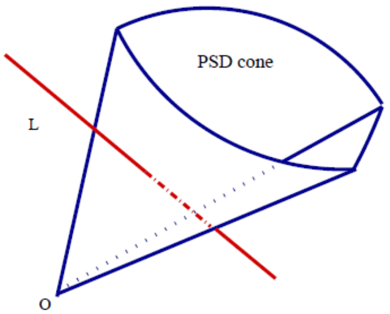
$$\begin{aligned} p(x) &= (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 \\ &\quad +(4x_2^2 - x_3^2)^2. \end{aligned}$$

Natural questions:

- **Q1:** Is it any easier to test for a sum of squares (SOS) decomposition?
- **Q2:** Is every nonnegative polynomial SOS?

Sum of squares and semidefinite programming

[Lasserre], [Nesterov], [Parrilo]



Q1: Is it any easier to decide SOS?

- Yes! Can be reduced to a **semidefinite program (SDP)**

- Can also efficiently **search and optimize** over SOS polynomials

- As we will see, this latter property is very important in applications…

Semidefinite programming (SDP)

- A broad generalization of linear programs
- Can be solved to arbitrary accuracy in polynomial time (e.g., using interior point algorithms) **[Nesterov, Nemirovski], [Alizadeh]**

$$\begin{aligned} \min_{X \in S^{n\times n}} \quad & \mathrm{Tr}(CX) \\ \text{s.t.} \quad & \mathrm{Tr}(A_i X) = b_i \quad i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

"psd"

Notes: $\mathrm{Tr}(CX) = \sum_{i,j} C_{ij} X_{ij}$

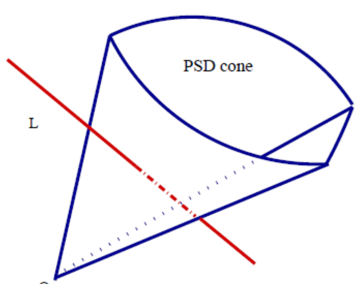
$X \succeq 0: \; y^T X y \geq 0 \quad \forall y \in \mathbb{R}^n$

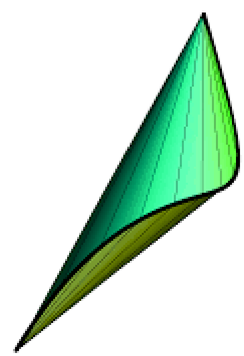
$\Updownarrow$

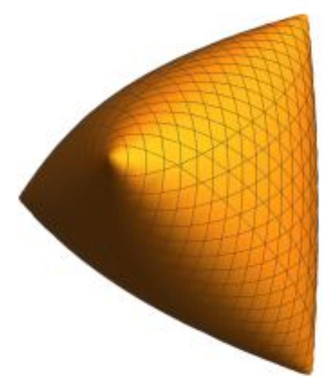
Eigenvalues of X are ≥ 0.

Data to SDP: $C, A_1, \dots, A_m \in S^{n\times n}, b_1, \dots, b_m \in \mathbb{R}$

Feasible set called a "spectrahedron":







SOS→SDP

Thm:

A polynomial p of degree $2d$ is SOS if and only if $\exists Q \succcurlyeq 0$ such that

$$p(x) = z(x)^T Q z(x)$$

where $z = \left[1, x_1, \dots, x_n, x_1x_2, \dots, x_n^d\right]^T$ is the vector of monomials of degree up to d.

(It follows that checking membership or optimizing a linear function over the set of SOS polynomials is an SDP)

Proof: $(\Rightarrow)$ Suppose $\exists Q \succcurlyeq 0$ s.t. $p(x) = z^T(x) Q z(x) \ \forall x$.

$$Q \succcurlyeq 0 \Rightarrow Q = \underset{\substack{\downarrow \\ r \times \binom{n+d}{d}}}{V^T V} \Rightarrow p(x) = z^T(x) V^T V z(x) = \|V z(x)\|^2 = \sum_{i=1}^{r} \left(v_i^T z(x)\right)^2.$$

$(\Leftarrow)$ Suppose $p(x)$ is SOS.

$$\exists v_1, \dots, v_r \in \mathbb{R}^{\binom{n+d}{d}} \text{ s.t. } p(x) = \sum_{i=1}^{r} \left(v_i^T z(x)\right)^2 = \sum_{i=1}^{r} \left(z^T(x)\, v_i\right)\left(v_i^T z(x)\right) = z^T(x) \overbrace{\left(\sum_{i=1}^{r} v_i v_i^T\right)}^{:= Q} z(x).$$

Example

$$\boxed{p(x) = 10x^4 - 2x^3 - 7x^2 + 4x + 4}$$

Is p SOS ?

SDP feasibility problem

$$p(x) = \underbrace{\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T}_{z^T(x)} \underbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}}_{z(x)}$$

$\forall x$

Find $Q \succeq 0$ s.t.

$q_{33} = 10, \quad 2q_{23} = -2$

$q_{22} + 2q_{13} = -7$

$2q_{12} = 4, \quad q_{11} = 4$

SDP solver output:

$$Q = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \overset{\text{e.g. Cholesky}}{=} \underbrace{\begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix}}_{V^T} \underbrace{\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}}_{V}$$

$$\Rightarrow p(x) = z^T(x) V^T V z(x) = \| V z(x) \|^2 = \left\| \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \right\|^2$$

$$\boxed{p(x) = (2x + x^2)^2 + (2 + x - 3x^2)^2}$$

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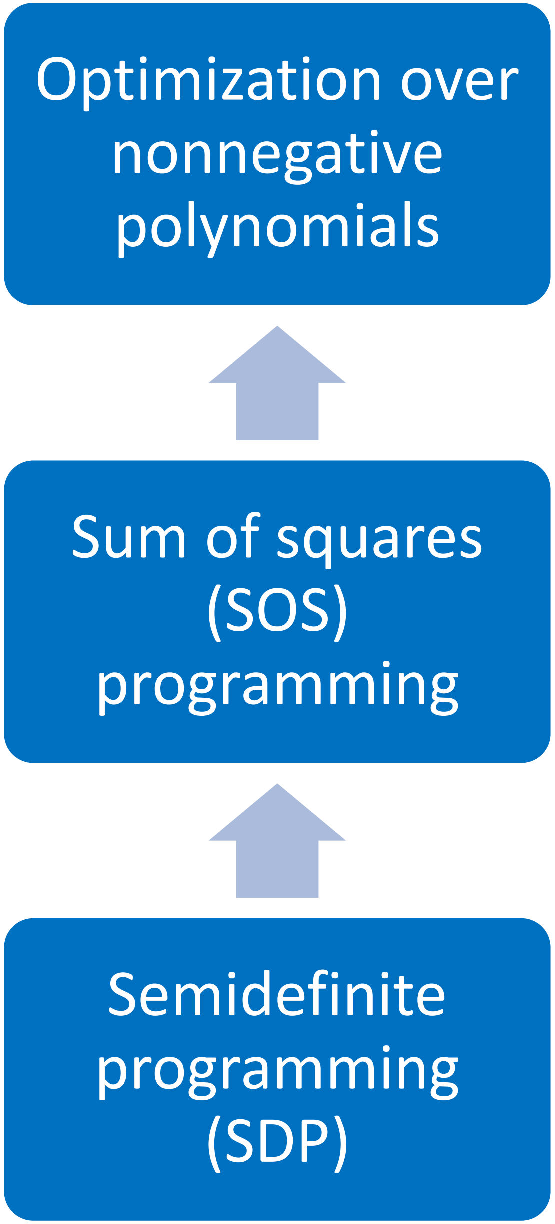
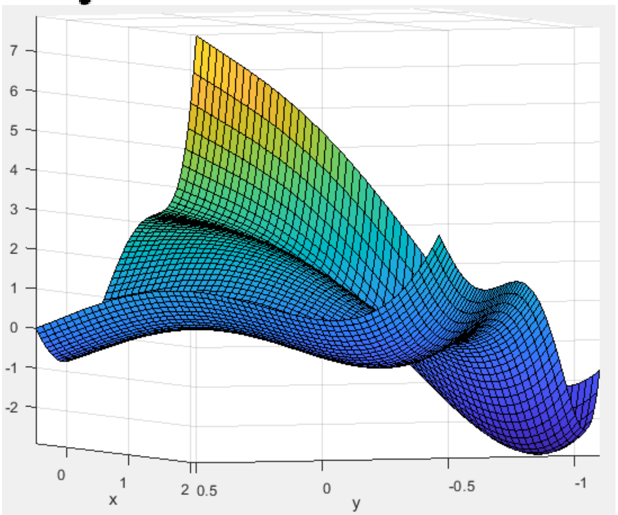
Let's revisit two of our applications!


Optimization over nonnegative polynomials
Sum of squares (SOS) programming
Semidefinite programming (SDP)

1) Nonconvex unconstrained minimization

Find: $p^* := \inf_{(x,y)\in\mathbb{R}^2} 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4 + x^2y$ ← $p(x,y)$



$$\boxed{\begin{aligned} p_{sos} := \sup_{\gamma\in\mathbb{R}} \quad & \gamma \\ \text{s.t.} \quad & p(x,y) - \gamma \quad \text{sos} \end{aligned}}$$

→ SDP!

$p_{sos} \leq p^*$

```
p=4*x^2-2.1*x^4+(1/3)*x^6+1*x*y-4*y^2+4*y^4+x^2*y;
solvesos(sos(p-gam),-gam,[],[gam])
p_sos=double(gam)
```

solvertime: 0.6 (s)

p_sos =

-2.92156095096358

```
[inf,z,Q]=solvesos(p-p_sos);
sdisplay(z{1})
[v,d]=eig(double(Q{1}));
zxstar=v(:,1)/v(1,1);
xstar=[zxstar(3);zxstar(2)]
p_at_xstar=replace(p,[x,y],[xstar(1),xstar(2)])
```

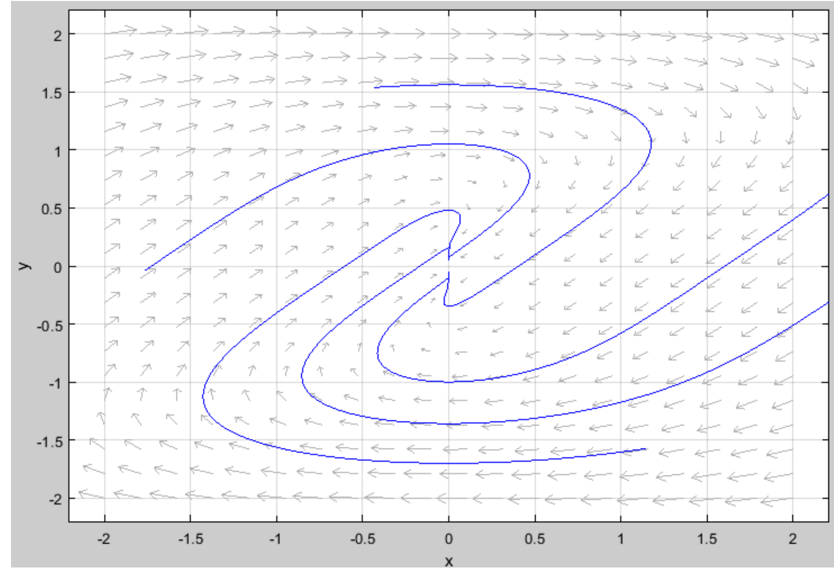
```
xstar =

   1.832996144755618
  -0.92293147842127

p_at_xstar =

  -2.921559422066406
```

2) Automated proof of global asymptotic stability



MK [redacted].edu> on behalf of
[redacted]

Tue 2/9/2021 1:13 PM
To: Amir Ali Ahmadi

Hi Amir Ali,

I hope life and career are going well.

I have a question that I assume might take little more than 5-10 of your time but please feel free to let me know if it would actually take more.

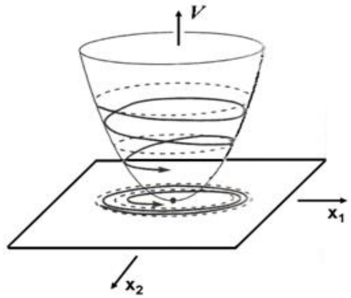
> Today in class we got into an interesting discussion with students about what a strict Lyapunov function would be for the system
>
> dx/dt = -x +y^3
> dy/dt = -x
>
> A non-strict L.f.. is easy, V = x^2/2 + y^4/4, with dV/dt = - x^2. One could then deduce g.a.s. by a Barbashin-Krasovskii/Lasalle argument, but that's not satisfactory.
>
> I started constructing a strict one in real time and it quickly got out of hand, necessitating higher and higher powers and many cross terms. I inevitably thought of you and your (and Pablo's) SOS program that would spit out a good strict V within seconds.

If you can plug in this system and let me know what comes out, I'd appreciate it, and my 40-50 students in class would learn a few things (complexity of Lyapunov functions, automated options for finding them, etc.).

Best regards,
[redacted]

Automated proof of global asymptotic stability

$$\begin{bmatrix} \dot{x} = -x + y^3 \\ \dot{y} = -x \end{bmatrix}$$



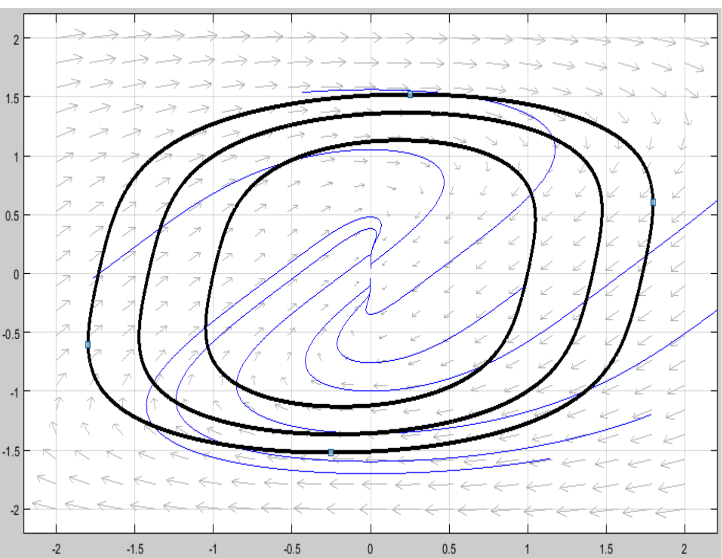
$$\boxed{\begin{array}{l} \text{Find } V(x,y) \text{ of degree 4 s.t.} \\ \qquad V(x,y) \quad \text{SOS} \\ \qquad -\dot{V}(x,y) = -\left\langle \nabla V(x,y), \begin{pmatrix} -x+y^3 \\ -x \end{pmatrix} \right\rangle \quad \text{SOS} \end{array}} \rightarrow \text{SDP!}$$

```
sdpvar x y
xdot=-x+y^3;
ydot=-x;

[V,c,m]=polynomial([x;y],4,2);
Vdot=jacobian(V,[x,y])*[xdot;ydot];
FF=[sos(V),sos(-Vdot)]
solvesos(FF,[],[],[c])
```

```
>> sdisplay(clean(double(c)'*m,1e-3))
1.00000084865*x^2-0.333330248293*x*y+0.166665124147*y^2+0.500118639025*y^4
```

$$V(x,y) = x^2 - \frac{1}{3}xy + \frac{1}{6}y^2 + \frac{1}{2}y^4$$



$$= \left(x - \frac{1}{6}y\right)^2 + \frac{5}{36}y^2 + \frac{1}{2}y^4 \qquad \text{(hence positive definite)}$$

$$\geq \frac{1}{36}(x^2+y^2) + \frac{1}{2}y^4 \qquad \text{(hence radially unbounded)}$$

$$\dot{V}(x,y) = -\frac{5}{3}x^2 - \frac{1}{3}y^4 \qquad \text{(hence negative definite)}$$

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Hilbert's 1888 Paper

Q2: SOS $\overset{?}{\Leftarrow}$ Nonnegativity

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

From Logicomix

Motzkin (1967):

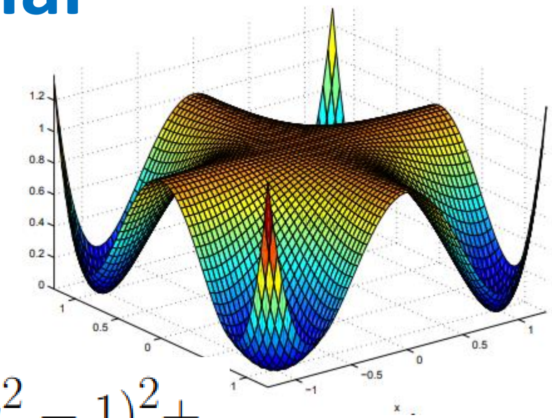
$$M(x_1,x_2) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$$

Robinson (1973):

$$R(x_1,x_2,x_3) = x_1^2(x_1-1)^2 + x_2^2(x_2-1)^2 + x_3^2(x_3-1)^2 + 2x_1x_2x_3(x_1+x_2+x_3-2)$$

The Motzkin polynomial

$$M(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$

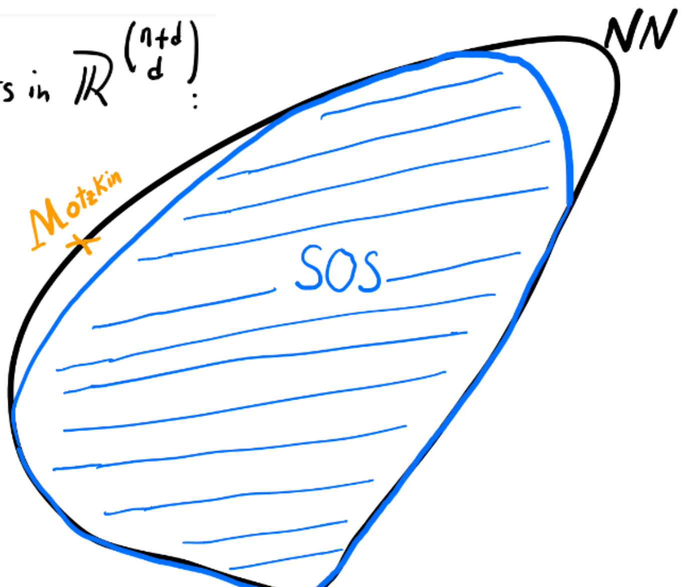


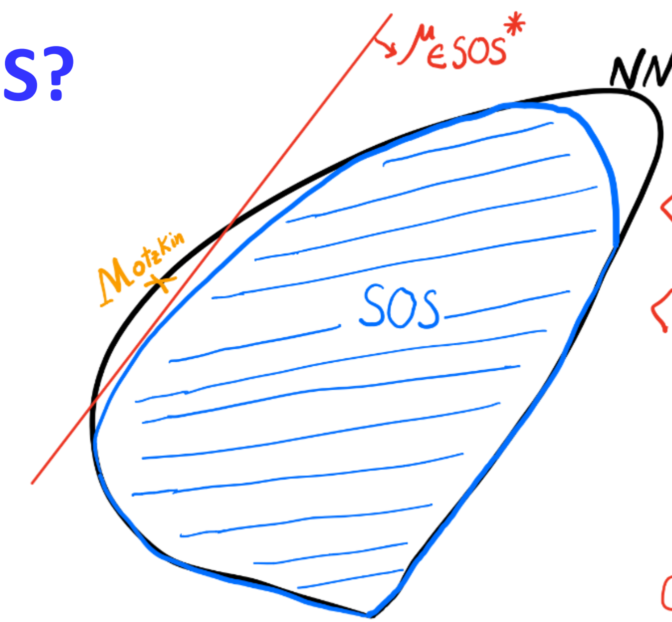
How to prove it is nonnegative?

$$(x^2+y^2+1)\,M(x,y) = (x^2y-y)^2 + (xy^2-x)^2 + (x^2y^2-1)^2 + \frac{1}{4}(xy^3-x^3y)^2 + \frac{3}{4}(xy^3+x^3y-2xy)^2$$

How to prove it is not SOS?

Two sets in $\mathbb{R}^{\binom{n+d}{d}}$:





$\langle \mu, \mathrm{coeff}(M) \rangle < 0$

$\langle \mu, \mathrm{coeff}(q) \rangle \geq 0 \;\; \forall q \in SOS$

$\Updownarrow$

$\left[z(x) z(x)^T \right]\Big|_{\mu} \succeq 0$

Can find μ with SDP.

$NN^* \subseteq SOS^*$

Hilbert's 17th Problem (1900)

Q. p **nonnegative** $\overset{?}{\Rightarrow}$ $$p = \sum_i \left(\frac{g_i}{q_i}\right)^2$$

- Artin (1927): **Yes!**

- Implications:
  - $p \geq 0 \Rightarrow \exists \boldsymbol{h}$ **sos** such that $\boldsymbol{p}.\,\boldsymbol{h}$ **sos**
  - **Reznick:** (under mild conditions) can take $\boldsymbol{h} = \left(\sum_i \boldsymbol{x}_i^2\right)^{\boldsymbol{r}}$
  - Certificates of nonnegativity can *always* be given with sos (i.e., with semidefinite programming)!
  - We'll see how the Positivstellensatz generalizes this even further…

Outline of the rest of the talk…

- **Global nonnegativity**
  - Sum of squares (SOS) and semidefinite programming
  - Two applications
  - Hilbert's 17th problem
- **Nonnegativity over a region**
  - Putinar's Positivstellensatz
  - Two applications
- **Recap and further reading**

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Positivstellensatz

$p(x) > 0, \forall x \in \mathbb{R}^n$

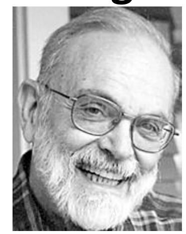
Artin

If $p(x) \geq 0, \forall x \in \mathbb{R}^n$,
then $\exists$ sos q s.t. $p \cdot q$ sos.

1927

20^{th} century

Stengle



1974

1991

1993

$p(x) > 0,$
$\forall x \in S = \{x \mid g_i(x) \geq 0\}$

Schmudgen

Putinar

Putinar's Positivstellensatz (1993)

$$\boldsymbol{p(x) \geq 0 \text{ on } S = \{x \in \mathbb{R}^n | g_i(x) \geq 0, i = 1, \dots, m\}}$$

easy direction $\Uparrow$ $\Downarrow$ (under mild assumptions)

$\exists$ SOS polynomials $s_0(x), \dots, s_m(x)$such that

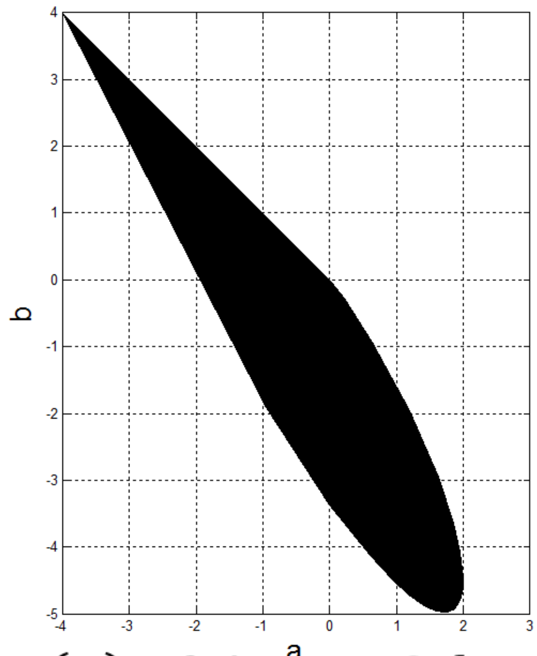
$$\boldsymbol{p(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x)}$$

- Search for SOS multipliers can be automated via SDP!
- This means that SDP-based proofs of implications between polynomial inequalities always possible!
- Degree bounds on SOS multipliers available in special cases.

How did I plot this?

- For what values of a, b is the following polynomial monotone over $[0,1]$?

$$p(x) = x^4 + ax^3 + bx^2 - (a+b)x$$



Theorem. A polynomial $p(x)$ of degree $2d$ is monotone on $[0,1]$ if and only if

$$p'(x) = xs_1(x) + (1-x)s_2(x),$$

where $s_1(x)$ and $s_2(x)$ are some SOS polynomials of degree $2d-2$.

Let’s end with a couple applications:

- **Finance**
- **Control**

Optimization over nonnegative polynomials

↑

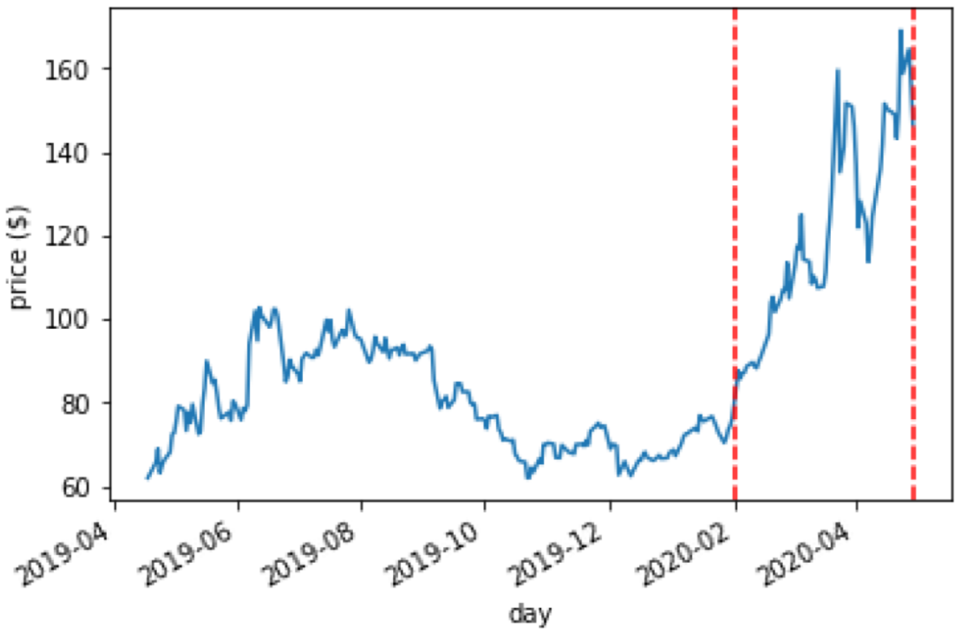
Sum of squares (SOS) programming

↑

Semidefinite programming (SDP)

Distributionally robust optimization

What's the probability that Zoom's stock goes bust?



- Three months starting Feb 1, 2020

$$r_i = \frac{P_i - P_{i-1}}{P_i}, \quad i = 1, \ldots, 61$$

- Empirical moments $m_k = \mathbb{E}[r^k]$:

$$m_1 = 0.0068, m_2 = 0.0034,$$
$$m_3 = 2 \times 10^{-6}, m_4 = 5 \times 10^{-5}$$

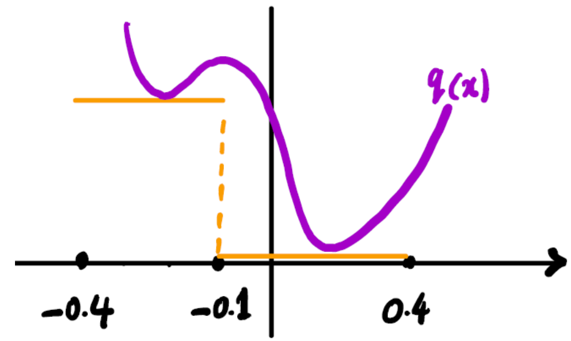
- The distribution of r is supported on [-0.4,0.4] but is otherwise unknown
- What is the probability that Zoom's stock return will be below -0.1 today?
- Want the worst-case probability over all distributions whose first 4 moments are within 10% of those computed from data.

Sum of squares optimization can compute this probability!

$$
\begin{aligned}
\alpha := \inf_{q,r,s,\gamma} \quad & \gamma \\
\text{s.t.} \quad & q(x) = \sum_{k=0}^{4} q_k x^k \ \text{ is a degree-4 (univariate) polynomial,} \\
& r(x), s(x) \ \text{ are quadratic polynomials that are sos,} \\
& q_0 + \sum_{k=1}^{4} q_k m_k' \leq \gamma \ \forall m_k' \in [0.9\, m_k, 1.1\, m_k] \text{ for } k = 1, \ldots, 4, \\
& q(x) - (0.4^2 - x^2)\, s(x) \text{ is sos,} \\
& q(x) - 1 - (0.4 + x)(-0.1 - x) r(x) \text{ is sos.}
\end{aligned}
$$

$\Longrightarrow q(x) \geq 0 \quad \forall x \in [-0.4, 0.4]$

$\Longrightarrow q(x) \geq 1 \quad \forall x \in [-0.4, -0.1]$

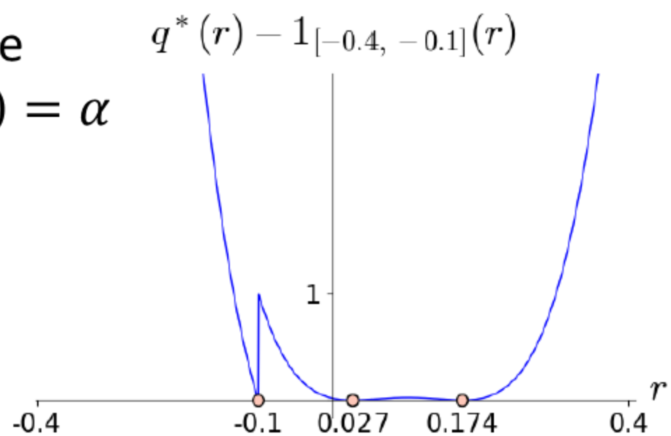


$\mathbb{P}(r \in [-0.4, -0.1]) = \mathbb{E}[1_{[-0.4,-0.1]}]$ $\Rightarrow 1_{[-0.4,-0.1]} \leq q(x) \ \forall x \in [-0.4, 0.4]$

$\Rightarrow \mathbb{E}[1_{[-0.4,-0.1]}] \leq \mathbb{E}[q(x)] = \sum_{k=0}^{4} q_k m_k \leq \gamma$

$\mathbb{P}(r \in [-0.4, -0.1]) \leq \alpha$

In fact, we always have
$\mathbb{P}(r \in [-0.4, -0.1]) = \alpha$




```
Optimizer terminated. Time: 0.17
alpha =

    0.2073
```

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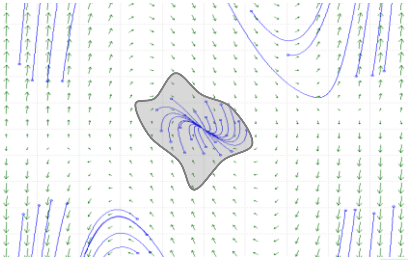
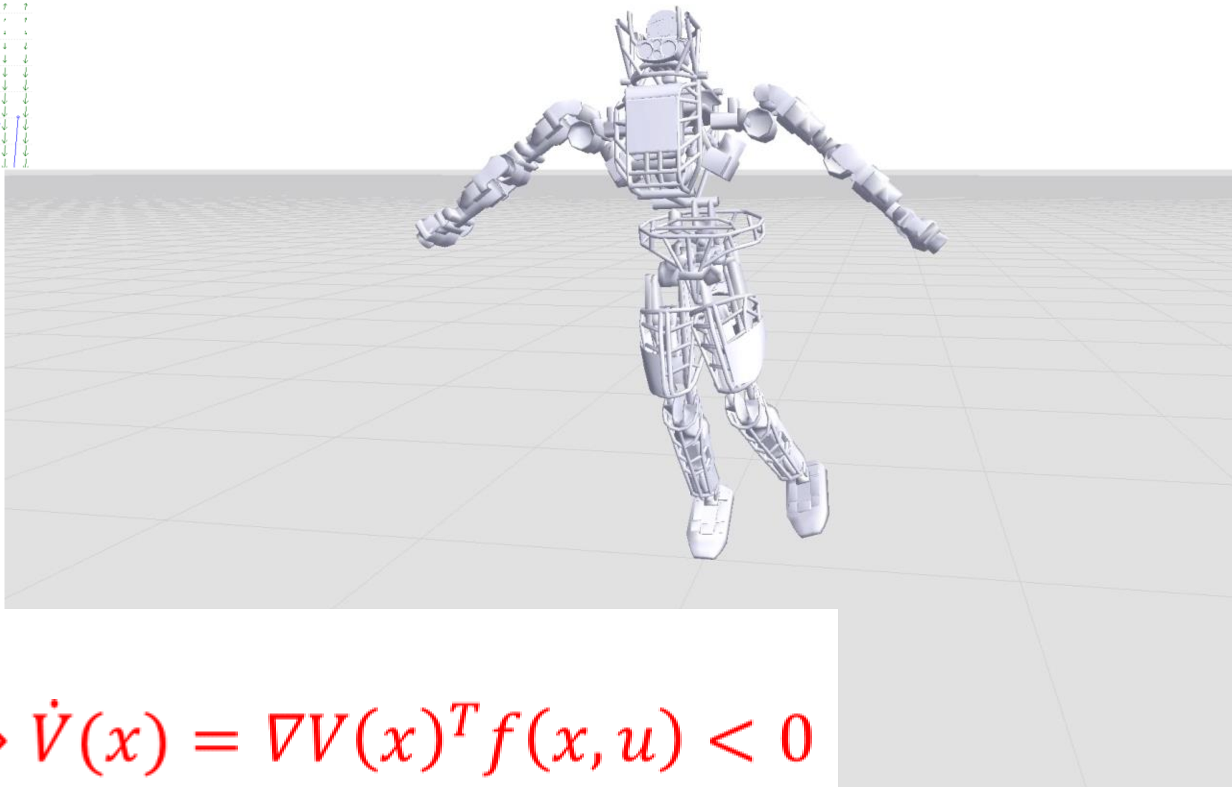
Stabilizing a humanoid robot on one foot

$\dot{x} = f(x, u)$ 30 states 14 control inputs Cubic dynamics





$$V(x) > 0$$
$$V(x) \le \beta \Rightarrow \dot{V}(x) = \nabla V(x)^T f(x, u) < 0$$

(w/ Majumdar and Tedrake)

Certifying collision avoidance

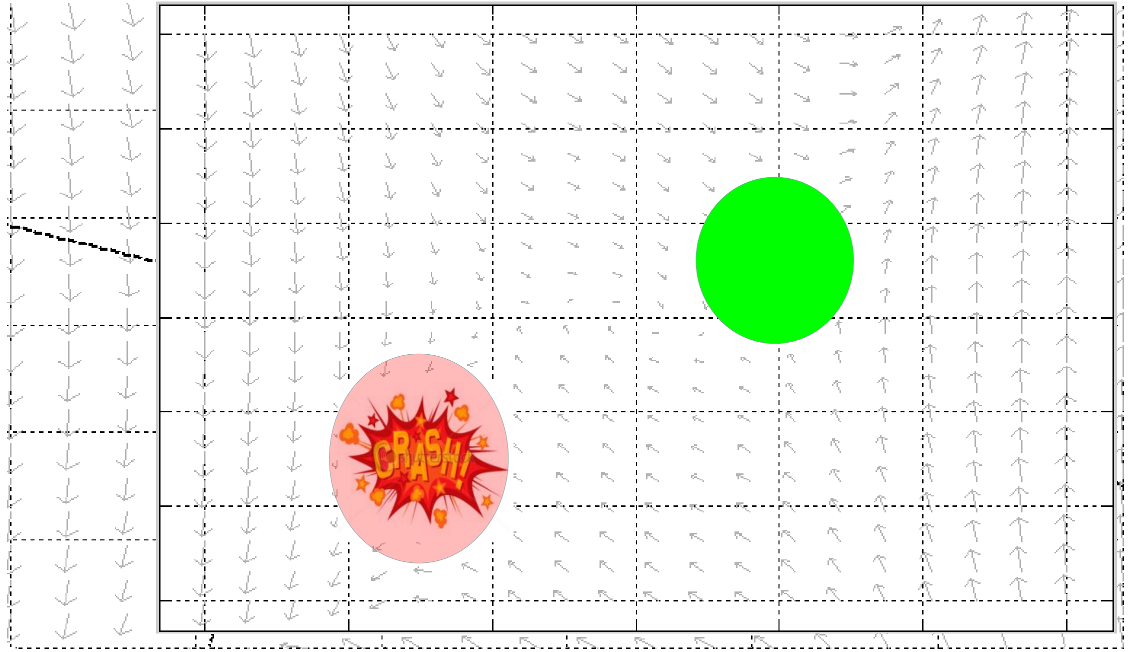
$$\dot{x} = f(x)$$

(vector valued polynomial)

$\mathcal{S}$: needs safety verification

$\mathcal{U}$: unsafe (or forbidden) set

(both sets basic semialgebraic)

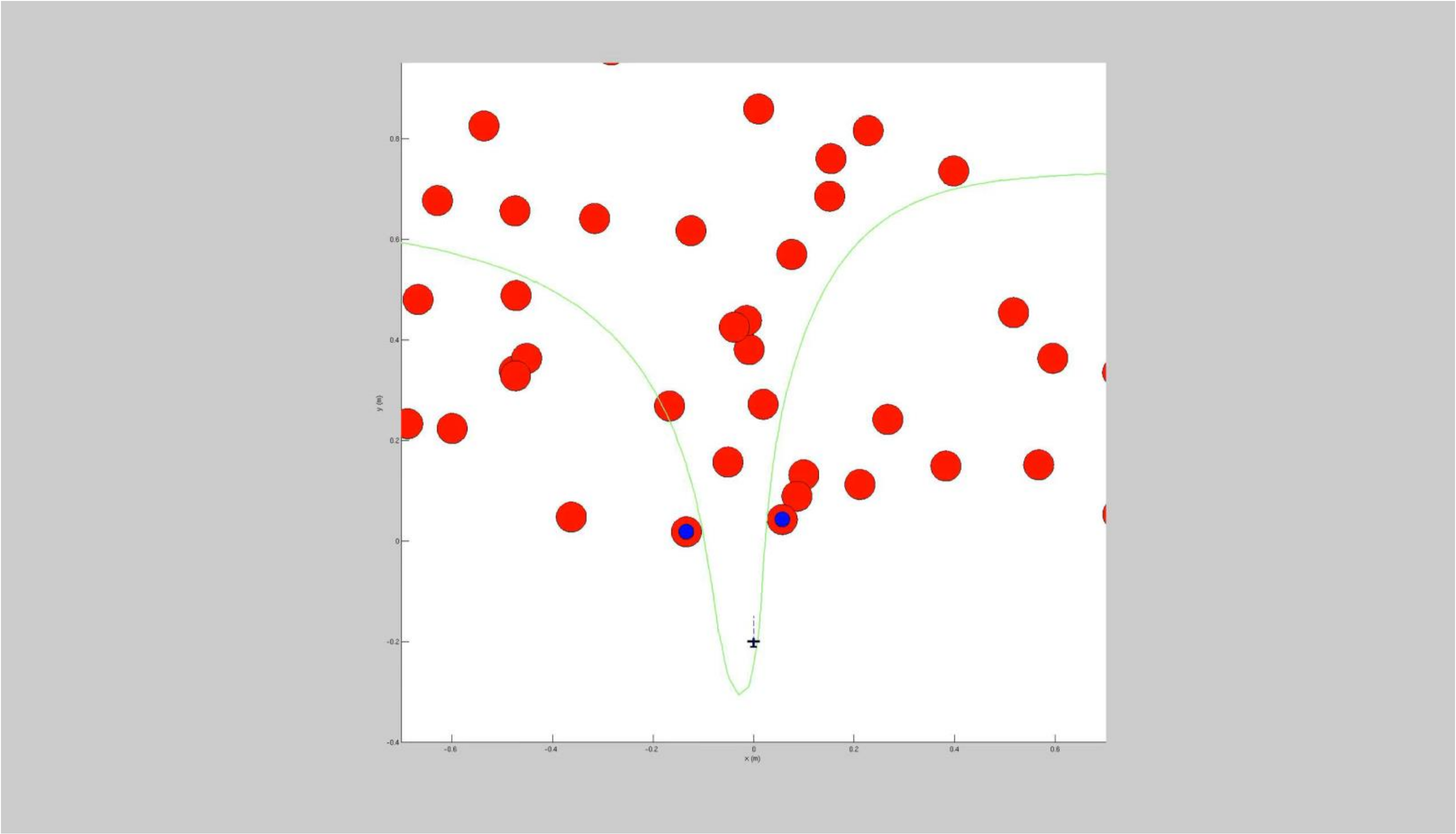


Safety guaranteed if we find a "Lyapunov function" such that:

$$B(\mathcal{S}) < 0$$
$$B(\mathcal{U}) > 0$$

$$\dot{B} = \langle \nabla B(x), f(x) \rangle \leq 0$$

Real-time collision avoidance certificates



(w/ Majumdar)

Dubins car model

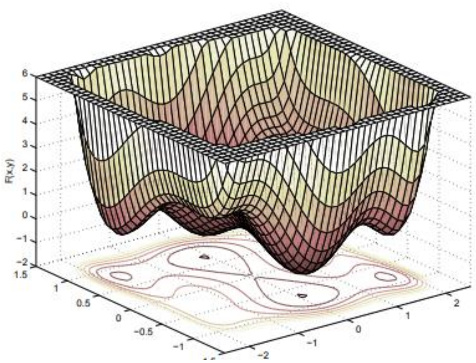
Run-time: 20 ms

Recap: “See an inequality? Think SOS!”

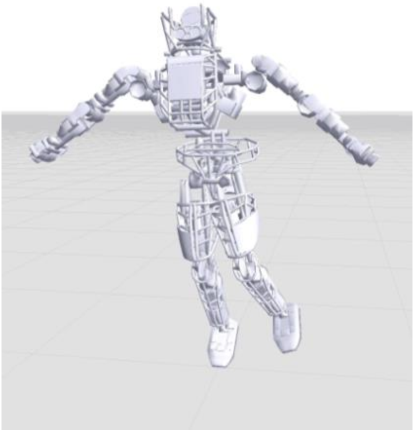
Is $p(x) \geq 0$ on $\{g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$?

Automated SOS-based proofs via SDP!

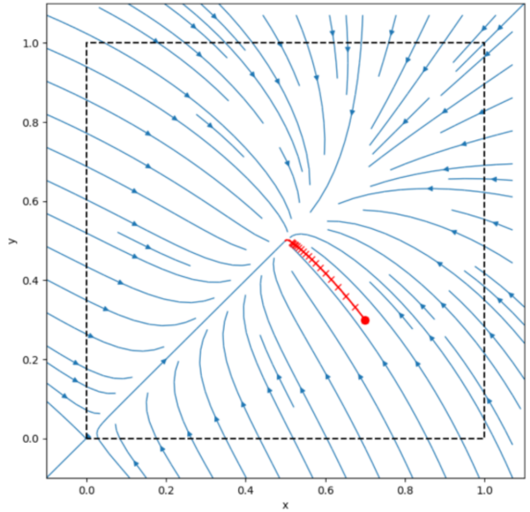
Many applications!



Optimization

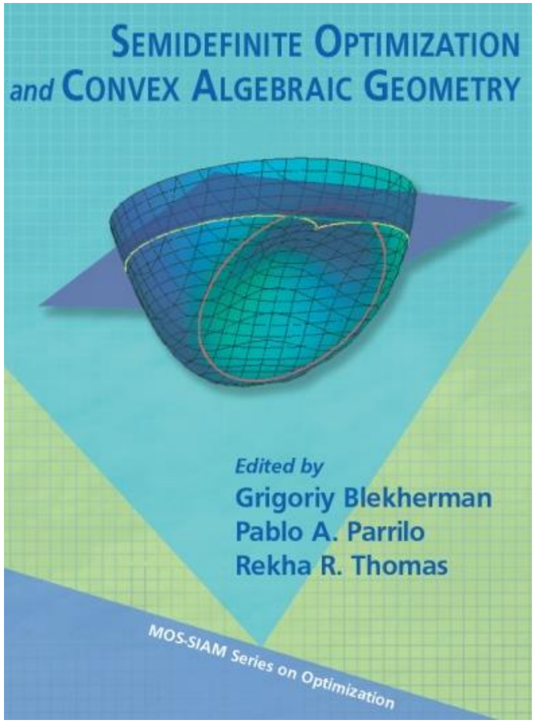


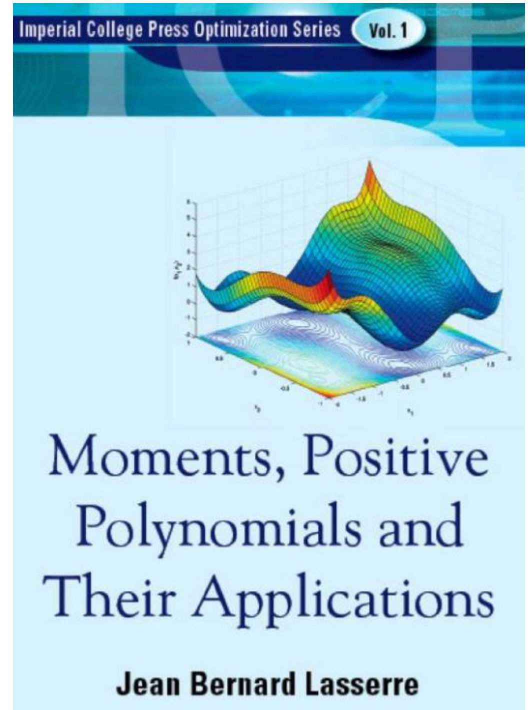
Control



Learning

Want to learn more?





SUMS OF SQUARES, MOMENT MATRICES AND OPTIMIZATION OVER POLYNOMIALS

MONIQUE LAURENT*

Applications of sums of squares

Georgina Hall