Princeton University

ORF 523
Final Exam, Spring 2024

48-hour period within May 8, 8AM - May 15, 10PM EDT

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1. Please write your name on the first page of your solutions. Next to it, please write out and sign the following pledge: "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this examination."
2. The exam is not to be discussed with *anyone* except possibly the instructors and the AIs. You can only ask *clarification questions*, and only as *public* (and preferably non-anonymous) questions on Ed Discussion. No emails.
3. You are allowed to consult the lecture notes, your own notes, the reference books of the course as indicated on the syllabus, the problem sets and their solutions (yours and ours), the midterm and its solutions (yours and ours), the practice midterm and final exams and their solutions, all Ed Discussion posts, but *nothing else*. You can only use the Internet in case you run into problems related to software.
4. You may refer to facts stated in the notes or problem sets without proof.
5. For computational problems, include your code. The output you present should come from your code. Report requested numerical values to 4 digits after the decimal point.
6. You have 48 hours from the time of download to submit this exam on Gradescope as a *single PDF file*. The latest submission time is Wednesday (May 15, 2024) at 10PM EDT. You are free to write your solutions on paper or on a tablet, or to type them up. Only the latest version submitted before your deadline will be graded.
7. Each question has 25 points. You need to justify your answers to receive full credit.

Problem 1
Suppose a set $S \subseteq \mathbb{R}^n$ is nonempty, convex, and closed and that its complement $\bar{S}$, i.e., the set $\mathbb{R}^n \backslash S$, is convex and nonempty. Show that S must be a halfspace.

Problem 2
For a graph G on n vertices, let $\alpha(G)$ be the stability number of G, $\vartheta(G)$ be the Lovász theta number of G, and $\vartheta'(G)$ be the optimal value of the semidefinite program:

$$\begin{aligned} \vartheta'(G) = \min_{P \in S^{n\times n}, k\in\mathbb{R}} \quad & k \\ \text{s.t.} \quad & k(I+A) - J - P \geq 0 \\ & P \succeq 0. \end{aligned}$$

Here, A is the adjacency matrix of G, I is the identity matrix, J is the matrix of all ones, and the first inequality is entrywise.

(a) Show that $\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$ for all graphs G.

(b) Let G be the graph with vertex set corresponding to all the vectors in $\{0,1\}^6$, two distinct vectors being adjacent if their ℓ_1 distance (i.e., Hamming distance) is less than 4. For your convenience, the adjacency matrix of this graph is given in the file `AdjacencyMatrix.mat`.[1] Compute $\alpha(G)$, $\vartheta'(G)$, and $\vartheta(G)$.

Problem 3
Show that the following decision problem is NP-complete.
CONCAVE-BOX-QP: Given a symmetric matrix $Q \in \mathbb{Q}^{n\times n}$, with $Q \preceq 0$, vectors $c, l, u \in \mathbb{Q}^n$, and a scalar $k \in \mathbb{Q}$, decide whether the optimal value of the following optimization problem is less than or equal to k:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & -l_i \leq x_i \leq u_i \quad i = 1, \ldots, n. \end{aligned}$$

[1]Download:`https://www.princeton.edu/~aaa/Public/Teaching/ORF523/S24/AdjacencyMatrix.mat`

Problem 4

A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ is *separable* if it can be written as $p(x) = \sum_{i=1}^{n} q_i(x_i)$, where each q_i is a univariate polynomial.

(a) Show that a separable polynomial is nonnegative if and only if it is a sum of squares. (You can use the fact that a univariate nonnegative polynomial is a sum of squares without proof.)

(b) Present an explicit family of degree-4 polynomials $p_n : \mathbb{R}^n \mapsto \mathbb{R}$ such that

(i) the number of nonglobal local minima of p_n grows exponentially with n,

(ii) for all n, we have

$$\begin{bmatrix} \min_{x \in \mathbb{R}^n} p_n(x) \end{bmatrix} = \begin{bmatrix} \max_{\gamma \in \mathbb{R}} & & \gamma \\ \text{s.t.} & p_n(x) - \gamma & \text{is a sum of squares} \end{bmatrix}.$$

For context, this means that semidefinite programming can find the exact optimal value of nonconvex problems with exponentially many spurious local minima.