ORF 523	PROBLEM SET 5	Spring 2025, Princeton University
Instructor: A.A. Ahmadi		
AIs: Y. Hua, J.J. Macosko		
Due on Apr 10, 2025, at 1:30pm ET, on Gradescope		

For all problems that involve coding, please include your code.

Problem 1: SDPs with rational data but no rational feasible solution
Give an example of symmetric $n \times n$ matrices $A_1, \ldots, A_m$ with rational entries and rational numbers $b_1, \ldots, b_m$ such that the set

$$S := \{X \in S^{n \times n} | \ \mathrm{Tr}(A_i X) = b_i, i = 1, \ldots, m, X \succeq 0\}$$

is non-empty, but only contains matrices that have at least one irrational entry. Here, $S^{n \times n}$ denotes the set of symmetric $n \times n$ matrices with real entries and Tr stands for the trace operation. You can pick any value for n and m that you like as long as the above requirements are met.[1]

Problem 2: Stability of a pair of matrices
Recall that the spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\rho(A)$, is the maximum of the absolute values of its eigenvalues. We call a matrix "stable" if $\rho(A) < 1$. Let us call a pair of real $n \times n$ matrices $\{A_1, A_2\}$ stable if $\rho(\Sigma) < 1$, for any finite product Σ out of A_1 and A_2. (For example, Σ could be $A_2 A_1, A_1 A_2, A_1 A_1 A_2 A_1$, and so on.)

1. Does stability of A_1 and A_2 imply stability of the pair $\{A_1, A_2\}$?
2. Prove (possibly using optimization) that the pair $\{A_1, A_2\}$ with

$$A_1 = \frac{1}{4}\begin{pmatrix} -1 & -1 \\ -4 & 0 \end{pmatrix}, A_2 = \frac{1}{4}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix}$$

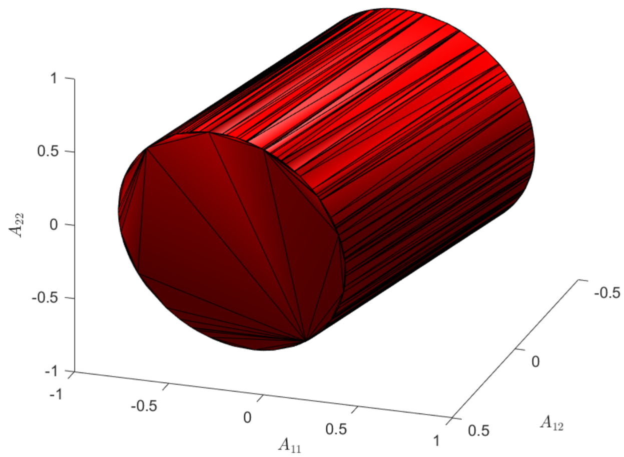
is stable.

[1]This exercise shows why it is in general difficult for an SDP solver to return an exact feasible solution. By contrast, the situation for LPs is much nicer as a feasible LP with rational data always has a rational feasible solution.

Problem 3: A nuclear program for peaceful reasons
The *nuclear norm* of a matrix $A \in \mathbb{R}^{m\times n}$ is given by

$$\|A\|_* := \sum_{i=1}^{\min\{m,n\}} \sigma_i(A),$$

where σ_i is the i-th singular value of A. The unit ball of this norm for symmetric 2×2 matrices is plotted below.



In optimization and machine learning, there is interest in the nuclear norm partly because it serves as the convex envelope of the function rank(A) over the set $\{A \in \mathbb{R}^{m\times n} |\ \|A\|_2 \leq\ 1\}$.[2] There are numerous application areas where one would like to minimize the rank of a matrix subject to affine constraints; examples include collaborative filtering or nonconvex quadratic programming.

1. Show that the dual norm of the spectral norm is the nuclear norm.

2. Show that the problem of minimizing the nuclear norm of a matrix subject to arbitrary affine constraints can be cast as a semidefinite program.

[2]What does this statement simplify to in the case where A is diagonal?

Problem 4: The Lovász sandwich theorem

The Lovász sandwich theorem states that for any graph $G(V, E)$, with $|V| = n$, we have

$$\alpha(G) \underset{(1)}{\leq} \vartheta(G) \underset{(2)}{\leq} \chi(\bar{G})$$

where

- $\alpha(G)$ is the stability number of G (i.e., the size of its largest independent set(s)),
- $\vartheta(G)$ is the Lovász theta number; i.e., the optimal value of the SDP

$$\begin{aligned} \vartheta(G) := \max_{X \in S^{n \times n}} \ & \mathrm{Tr}(JX) \\ \text{s.t.} \ & \mathrm{Tr}(X) = 1, \\ & X_{i,j} = 0, \ \text{if} \ \{i, j\} \in E \\ & X \succeq 0, \end{aligned}$$

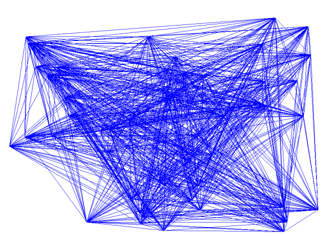
- $\chi(H)$ is the coloring number of H, that is the minimum number of colors needed to color the nodes of a graph H such that no two adjacent nodes get the same color, and
- $\bar{G}$ is the complement graph of G, i.e., a graph on the same node set which has an edge between two nodes if and only if G doesn't.

1. We proved inequality (1) in class. Prove inequality (2).
   Hint: You may want to first show that the optimal value of the following SDP also gives $\vartheta(G)$:

$$\begin{aligned} \min_{Z \in S^{(n+1)\times(n+1)}} \ & Z_{n+1,n+1} \\ \text{s.t.} \ & Z_{n+1,i} = Z_{ii} = 1, \ i = 1, \ldots, n \\ & Z_{ij} = 0 \text{ if } \{i, j\} \in \bar{E} \\ & Z \succeq 0. \end{aligned}$$

2. Given an example of a graph G where neither inequality (1) nor inequality (2) is tight.

Problem 5: Comparison of LP and SDP relaxations



For a graph $G(V, E)$, with $|V| = n$, we saw in class that an SDP-based upperbound for the stability number $\alpha(G)$ of the graph is given by $\vartheta(G)$ (as defined in Problem 1). We also saw that alternative upperbounds on the stability number can be obtained through the following family of LP relaxations:

$$
\begin{aligned}
\eta_{LP}^k := \max & \sum_{i=1}^n x_i \\
\text{s.t. } & 0 \le x_i \le 1,\ i = 1, \dots, n \\
& C_2 \dots, C_k,
\end{aligned}
$$

where C_k contains all clique inequalities of order k, i.e. the constraints

$$x_{i_1} + \dots + x_{i_k} \le 1$$

for all $\{i_1, \dots, i_k\} \in V$ defining a clique of size k.

1. Show that for any graph G, we have $\vartheta(G) \le \eta_{LP}^k\ \forall k \ge 2$.
   Hint: You may want to show that $\vartheta(G)$ can also be obtained as the optimal value of the following optimization problem:

$$
\begin{aligned}
\max_{Y \in S^{(n+1)\times(n+1)}} & \sum_{i=1}^n Y_{ii} \\
\text{s.t. } & Y \succeq 0, \\
& Y_{n+1,n+1} = 1, \\
& Y_{n+1,i} = Y_{ii},\ i \in V, \\
& Y_{ij} = 0\ ,\ \text{if } (i, j) \in E.
\end{aligned}
$$

2. The file `Graph.mat` contains the adjacency matrix of a graph G with 50 nodes (depicted above). Compute $\vartheta(G)$, η_{LP}^2, η_{LP}^3, η_{LP}^4 and $\alpha(G)$ for this graph. You can directly load the data file in MATLAB. In Python, you can use the following code to do this.

```
import scipy
mat = scipy.io.loadmat('Graph.mat')
G = mat['G']
```

3. Present a stable set of maximum size. Prove or disprove the claim that this graph has a unique maximum stable set.

4. What is the Shannon capacity of the graph G given in the data file?