Princeton University

ORF 523

Midterm Exam, Spring 2025

March 6, 2025, 1:30PM - 2:50PM EST.

Instructor:

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Please read the exam rules below before you start.

1. Please write your names on the exam booklet and on the exam sheet. Please return both items to us once the exam is over.
2. All questions should be answered in the booklet. Please write out and sign the following pledge on the booklet: "I pledge my honor that I have not violated the honor code during this examination."
3. You are allowed a single sheet of A4 paper, double sided, hand-written or typed.
4. No electronic devices are allowed (e.g., cell phones, calculators, laptops, etc.), except for checking the time.
5. You can cite results proven in lecture or on problem sets without proof.
6. Good luck!

You need to justify your arguments (correct proofs or counterexamples) to receive full credit.

Problem 1: Polytopes defined by totally unimodular matrices
Let $A \in \mathbb{R}^{m\times n}$ be a totally unimodular matrix and $b \in \mathbb{Z}^m$ be an integral vector. Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a polytope, i.e., a bounded polyhedron. Let $S = P \cap \mathbb{Z}^n$ be the set of integral points in P, and let $\text{conv}(S)$ denote the convex hull of S. Show that $P = \text{conv}(S)$.

Problem 2: A characterization of descent directions
Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex differentiable function. Show that a direction $d \in \mathbb{R}^n$ is a descent direction for f at a point $x \in \mathbb{R}^n$ if and only if $d^T \nabla f(x) < 0$. Furthermore, give an example to demonstrate that this statement is not true without the convexity assumption.

Problem 3: Closed convex sets
Let Ω be a non-empty closed convex set in $\mathbb{R}^n$. Show that for any point $y \in \mathbb{R}^n$, there exists a unique point $x^* \in \Omega$ that satisfies

$$(x - x^*)^T(y - x^*) \leq 0, \forall x \in \Omega.$$

Problem 4: Convex piecewise affine/quadratic functions
Let $\Omega_1, \ldots, \Omega_m \subseteq \mathbb{R}^n$ be non-overlapping sets with non-empty interior (i.e., each containing a ball of positive radius) such that $\mathbb{R}^n = \bigcup_{k=1}^m \Omega_k$. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be defined piecewise as

$$f(x) = f_k(x) \quad \text{when } x \in \Omega_k,$$

where $f_1, \ldots, f_m : \mathbb{R}^n \mapsto \mathbb{R}$ are given functions. In the two situations below, prove or disprove the following claim:

$$\text{“If } f \text{ is convex, then } f(x) = \max_{k\in\{1,\ldots,m\}} f_k(x), \forall x \in \mathbb{R}^n.\text{”}$$

(a) f is piecewise affine; i.e., for $k = 1, \ldots, m$, $f_k(x) = a_k^T x + b_k$, for some $a_k \in \mathbb{R}^n$ and $b_k \in \mathbb{R}$.

(b) f is piecewise quadratic; i.e., for $k = 1, \ldots, m$, $f_k(x) = x^T Q_k x + c_k^T x + d_k$, for some $Q_k \in \mathbb{R}^{n\times n}$, $c_k \in \mathbb{R}^n$, and $d_k \in \mathbb{R}$.