ELE 301, Fall 2010
Laboratory No. 6
Frequency Response of Continuous Time Systems

1 Background

1.1 Circuits in the Frequency Domain

For LTI continuous-time, bibo-stable systems, if the input is the sinusoid $e^{i\omega t}$, then the output is the sinusoid $\hat{h}(\omega)e^{i\omega t}$. The complex valued function of the (real valued) frequency variable ω is called the frequency response function, or just frequency response for short.

In the course notes, the frequency response of the RC circuit shown in Figure 1 was calculated to be

$$\hat{h}(\omega) = \frac{1}{1 + i\omega RC}.$$

This was obtained using the formula for the frequency response in terms of the impulse response:

$$\hat{h}(\omega) = \int_{\infty}^{\infty} h(t)e^{-i\omega t}dt$$

We now describe another method to get to this result. First, remember that we model resistors, capacitors and inductors as linear time-invariant components. To obtain the frequency response of a circuit made up of these components, we first look at each individual component's frequency response.

The current voltage relationship for a capacitance C is $I(t) = C(dV(t)/dt)$. So when $V(t) = Ve^{i\omega t}$, where $V = V_0 e^{i\phi}$, we have

$$I(t) = C(i\omega)Ve^{i\omega t} = \frac{Ve^{i\omega t}}{1/i\omega C} = \frac{V(t)}{1/i\omega C}.$$

So $I(t) = V(t)/Z_C$, where $Z_C = 1/i\omega C$.

If we want to do this for real valued sinusoids we would set $V(t) = V_0 \cos(\omega t + \phi)$, and obtain $I(t) = -C\omega \sin(\omega t + \phi)$. So the amplitude is scaled by $C\omega$ and there is a phase shift of $\pi/2$. Generally, it is much more convenient to use the complex sinusoids, because amplitude and phase can be represented by a single complex number multiplying the sinusoid.

Now let us look at an inductance L. The relationship between voltage and current is $V(t) = L(dI(t)/dt)$. As above consider $V(t) = Ve^{i\omega t}$ where $V = V_0 e^{i\phi}$ is a complex number giving the amplitude and phase. Then

$$I(t) = L(i\omega)Ve^{i\omega t} = \frac{Ve^{i\omega t}}{i\omega L} = \frac{V(t)}{i\omega L}$$

So $I(t) = V(t)/Z_L$, where $Z_L = i\omega L$.

You can apply the same analysis to a resistance R, and you will see that for $V(t) = Ve^{i\omega t}$, the current is $I(t) = V(t)/R$.

So for inputs that are complex exponentials, our circuit components have voltage-current relationships that look like Ohm's law, except that the impedances are now complex and in the case of capacitors and inductors are frequency dependent:

$$Z_R = R \qquad Z_C = 1/(i\omega) \qquad Z_L = i\omega L$$

All the rules of resistor combination in series and in parallel apply to impedances, so sinusoidal circuit analysis becomes very easy. We can also drop the $e^{i\omega t}$ term in the input during circuit calculation and then reinsert it at the end.

As an example, we analyze the RC circuit in Figure 1. Let $v_{in}(t) = Ve^{i\omega t}$. We know

Figure 1: Left: An RC filter. Right: The RC filter in the frequency domain.

that $v_{out}(t) = \hat{h}(\omega)e^{i\omega t}$. Hence we drop the term $e^{i\omega t}$ and do all our calculations assuming $v_{in} = V = V_o e^{i\phi}$. We redraw our circuit components as impedances, as show on the right of Figure 1, and use basic circuit laws to calculate $v_{out} = \hat{h}(\omega)$. The impedance circuit is a voltage divider, so

$$v_{out} = v_{in}\frac{1/i\omega C}{R + 1/i\omega C} = v_{in}\frac{1}{1 + i\omega RC}$$

Reinserting the sinusoidal signal we obtain

$$v_{out}(t) = \frac{1}{1 + i\omega RC}Ve^{i\omega t}$$

and

$$\hat{h}(\omega) = \frac{1}{1 + i\omega RC}$$

1.2 Second Order Low Pass Filters

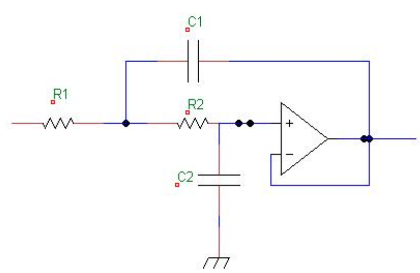

Figure 2: Active low pass filter: Sallen-Key circuit.

The circuit shown in Figure 2 is called a Sallen-Key circuit. The circuit contains two capacitors and gives rise to a second order differential equation. For this reason, it is called a second order filter. The frequency response of the circuit is:

$$\hat{h}(\omega) = \frac{1}{R_1 R_2 C_1 C_2 (i\omega)^2 + C_2(R_1 + R_2)(i\omega) + 1} \tag{1}$$

We can parameterize the frequency response of any second order low pass filter (e.g. the Sallen-Key circuit) as follows:

$$\hat{h}(\omega) = \frac{\omega_n^2}{(i\omega)^2 + 2\zeta\omega_n i\omega + \omega_n^2} \tag{2}$$

The two parameters ζ (zeta) and ω_n are called the *damping factor* and the *natural frequency* of the system, respectively. The fact that the numerator is a constant while the denominator is a quadratic in $i\omega$ indicates that the frequency response is low pass. The gain at $\omega = 0$ is 1 and as ω increases to infinity the gain asymptotically approaches zero. The constant in the numerator has been specifically chosen to normalize the gain to 1 at $\omega = 0$.

To find the maximum value of $|\hat{h}(\omega)|$ we minimize $(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2$ with respect to ω. This requires a simple exercise in differential calculus and yields:

- $|\hat{h}(\omega)|$ has a local maximum at $\omega = 0$; and
- if $\zeta < 1/\sqrt{2} \approx 0.7$, there is a second local maximum at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$.

Thus for small ζ the filter has a resonance at $\omega_r < \omega_n$. The gain G_r at the resonant frequency is obtained by substituting the expression for ω_r into the formula for $|\hat{h}(\omega)|$. This yields

$$G_r = (2\zeta\sqrt{1-\zeta^2})^{-1}.$$

The gain of the filter in dB is

$$\begin{aligned} G &= 20\log(\omega_n^2) - 10\log((\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2) \\ &= 20\log(\omega_n^2) - 10\log(\omega^4 - 2\omega^2(\omega_n^2 - 4\zeta^2) + \omega_n^4) \end{aligned} \tag{3}$$

For $\omega \gg \omega_n$ the ω^4 term dominates and the gain is approximately

$$G \approx 40\log(\omega_n) - 40\log(\omega)$$

So on the logarithmic frequency scale the Bode gain plot will be asymptotically linear with a slope of -40 dB per decade and the asymptote will intersect the 0dB line when ω equals the natural frequency ω_n.

When $\omega = 0$ the phase shift introduced by the filter is 0 degrees. This decreases as ω increases. At the resonant frequency ω_r the phase shift is $\tan^{-1}(\sqrt{1-2\zeta^2}/\zeta)$ degrees and at the natural frequency ω_n it is exactly -90 degrees. As ω goes to infinity the phase shift asymptotically approaches -180 degrees.

Suppose we want to select the values of R_1, R_2, C_1 and C_2 to give a low pass filter with a -3dB bandwidth of 100 rad/sec. Typically in a LP design you want to avoid any resonance. We can achieve this by seeting $\zeta^2 = 1/2$. Then $\omega_r = 0$ and the -3dB point is easily computed to occur at $\omega = \omega_n$. Now ω_n and ζ are known so there are now two constraint equations in three unknowns.

Typically you first fix a convenient value for C_1 and C_2, then solve the equations for appropriate values of R_1 and R_2. In practice we then have to do one additional step of selecting the nearest available values of the resistors from the standard resistor values, or use variable resistors and "tweak" them to get the desired values.

1.3 Bode Plots

It is common to display a frequency response function $\hat{h}(\omega)$, as a Bode diagram. As discussed in the notes, this consists of two plots, one of gain in dB vs. frequency and and one of phase vs. frequency, with the frequency axis in a logarithmic scale.

To do this, we first choose our lower and upper frequencies, for example 10^{-1} and 10^2 rad/sec. Then the code to create a bode plot would look something like this:

```
w=logspace(-1,2,200); %from 10^-1 to 10^2 with 200 points log spaced
mag=20*log10(abs(hh(w)));
phase=angle(hh(w))*180/pi;
figure;
subplot(2,1,1);semilogx(w,mag);set(gca,'xgrid','on','ygrid','on');
subplot(2,1,2);semilogx(w,phase);set(gca,'xgrid','on','ygrid','on') ;
```

where `hh(w)` is $\hat{h}(\omega)$. Obviously, for each subplot you will have to label the axes appropriately.

1.4 Filtering in MATLAB with `lsim`

Our filters operate on continuous time signals. To simulate the filter on a discrete time signals in MATLAB we use the function `lsim`. This requires requires expressing the frequency response function in terms of $s = i\omega$. If the frequency response is a rational function of s, then we can represent the numerator as a vector of coefficients `num` and the denominator as a vector of coefficients `den`. Then use the following line to construct the system in MATLAB:

```
H=tf(num,den);
```

Then we pass a signal `x`, sampled at time vector `t`, through `H` via `lsim`:

```
y=lsim(H,x,t);
```

1.5 Questions

1. Derive the frequency response of the circuit in Figure 3:

2. Derive the frequency response (1) of the Sallen-Key circuit (Figure 2):

2 Lab Procedure

2.1 Bode Plots

1. Plot a Bode plot of an ideal low pass filter with a cutoff frequency $B = 1$ rad/sec. Plot the same for an ideal high pass filter with cutoff frequency $B = 1$ rad/sec.

2. Next, plot the Bode diagram for the RC circuit in Figure 1 with $RC = 1$. This should match the Bode plot shown on page 4 of the "Eigenfunctions of LTI Systems" notes. Make sure all axes are appropriately labeled.

3. Plot the Bode diagram for the RC circuit in Figure 3 with $RC = 1$. What kind of filter is this? What is the approximate slope of the magnitude plot in the stop band in dB per decade?

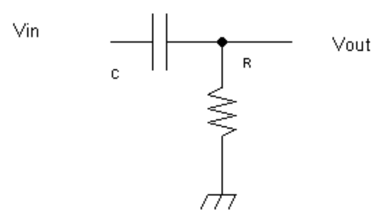

Figure 3: Another RC filter.

4. Suppose the output of the circuit in Figure 1 is the input of the circuit in Figure 3 (with appropriate isolation so the circuits do not interact). The frequency response of the combined system is the product of the frequency responses of each of the individual systems. Plot the Bode diagram of the cascaded system, with $RC = .1$ for the circuit in Figure 3 and $RC = .01$ for the low pass circuit. What kind of filter is this? What is the pass band? Does the ordering of the two circuits matter? (isolation is assumed.)

5. Plot the Bode diagram for the Sallen-Key circuit in Figure 2, with $R_1R_2C_1C_2 = 1$ and $C_2(R_1 + R_2) = .7654$. What kind of filter is this? What is the pass band? What is the approximate slope of the magnitude plot in the stop band in dB per decade? How does this filter compare to filter in 1?

2.2 Filtering Speech

We will now use the second order low pass filter to filter an audio file of speech summed with some high frequency tones. Write an m-file `spchfilt.m` that does the following:

1. - Choose ω_n such that the -3dB frequency is 600Hz and ζ such that there is no resonance.
   - Plot the Bode plot with frequency ranging from 100Hz to 10,000Hz.
   - Read the file whkight.wav, and plot the reconstructed signal vs time in a figure in subplot 1. Then play the signal with the `sound` command.
   - Finally, filter the reconstructed signal using the Sallen-Key circuit. You can do this with the `lsim` command. Plot the output vs time on figure 2 subplot 2, and play it.
2. Now lets add some unwanted noise and see if the filter can attenuate it without changing he speech too much. Just after you have reconstructed the speech signal but before you plot and play it, add a few new lines to your program that reads the sound file `sines.wav`, reconstructs the signal from its quantized version and then adds it to the speech. Now see if the filter can attenuate this corrupting signal without losing the speech in the process. Do the same with the sound file `hfnoise.wav`.

   In both cases you should note that the currupting signal is attenutated but not removed (why?). By moving the cut-off frequency of the filter around you can see the trade-off between getting better rejection of the noise and better preservation of the quality of the speech.

Demonstrate this program to the TA.

Attach all your plots and code to lab handout prior to handing it in.