Lecture 16

Thursday, November 17, 2011
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Corrections to last lecture : - OK to label "poles" and "zeros" even if $X(s)$ is not rational

- Consequence are not the same.

- Inverse Laplace Transform :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{\sigma+j\omega}\, d\omega$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\, ds$$

- Rational : $X(s) = \frac{N(s)}{D(s)}$

where $N(s)$ and $D(s)$ are polynomials.

Properties of the Laplace Transform:

Assume : $x(t) \xrightarrow{\mathcal{L}} X(s)$, $y(t) \xrightarrow{\mathcal{L}} Y(s)$

- Linearity : $a\,x(t) + b\,y(t) \xrightarrow{\mathcal{L}} a\,X(s) + b\,Y(s)$

ROC at least contains intercection.

- $x(t) = -y(t)$

$x(t) + y(t) \xrightarrow{\mathcal{L}} 0$ ROC $= \mathbb{C}$

- Time Shift : $x(t-t_0) \xrightarrow{\mathcal{L}} e^{-st_0}X(s)$, ROC doesn't change

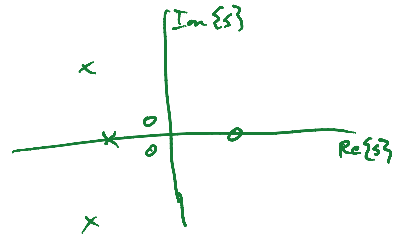
- Shift in s-domain : $e^{s_0 t}x(t) \xrightarrow{\mathcal{L}} X(s-s_0)$, ROC shift by s_0 $(R+s_0)$

- Time scaling : $x(at) \xrightarrow{\mathcal{L}} \frac{1}{|a|}X\left(\frac{s}{a}\right)$, ROC $= R_{new} = aR$

- **Conjugation:** $x^*(t) \xrightarrow{\mathcal{L}} X^*(s^*)$, ROC doesn't change.

If x is real, $x(t) = x^*(t)$

$\Rightarrow X(s) = X^*(s^*)$



- **Convolution:** $x(t) * y(t) \xrightarrow{\mathcal{L}} X(s)Y(s)$

ROC contains at least intesection

- ~~Multiplication~~

- ~~Duality~~

- **Differentiation:** $x'(t) \xrightarrow{\mathcal{L}} sX(s)$ ROC contains ROC of $X(s)$.

$\downarrow$ Opposite

- **Integration:** $\int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{L}} \frac{1}{s}X(s)$, ROC $= R \cap \{Re\{s\}>0\}$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$$

$$x(t) \xrightarrow{\mathcal{L}} X(s), \quad u(t) \xrightarrow{\mathcal{L}} 1/s$$

- **Diff. in s-domain:** $-t\,x(t) \xrightarrow{\mathcal{L}} X'(s)$, ROC doesn't change.

- **Initial Value Thm:** If $x(t)$ is causal and no $\delta(t)$.

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Real

– Final Value Thm: $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s X(s)$.

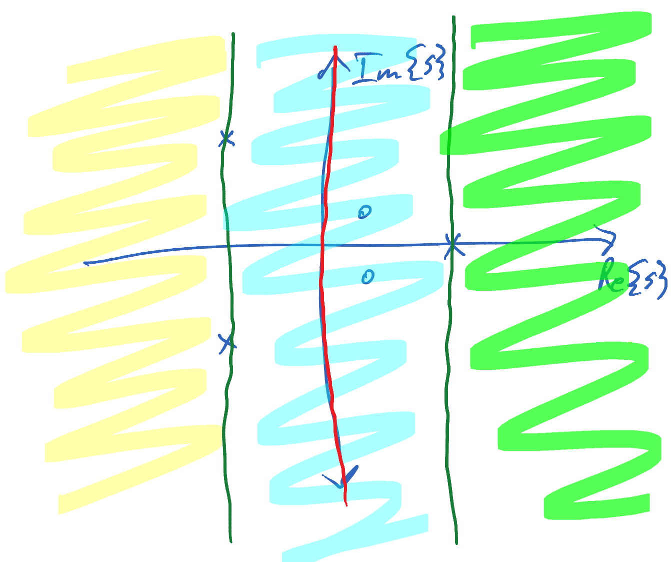
Stable system: (BIBO)

ROC contains the imaginary axis.
(of impulse response)

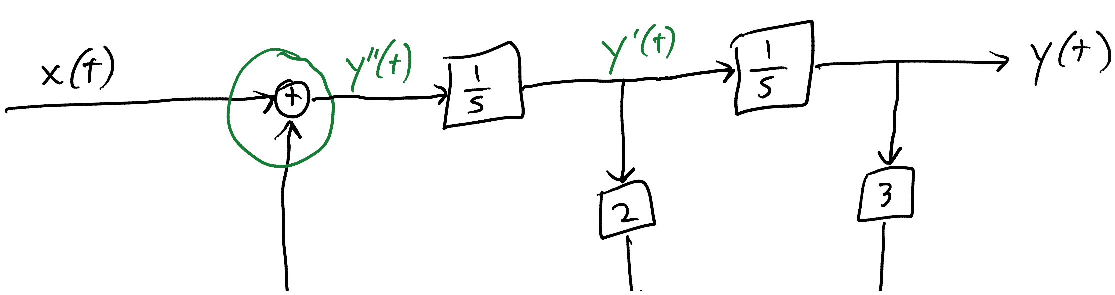
$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

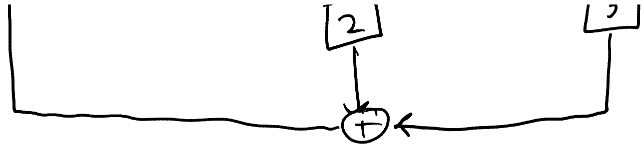
i.e. Fourier Transform exist. (necessary condition)

Causal: If $X(s)$ is rational, then causal $\Leftrightarrow$ right-sided.
$\Updownarrow$
ROC is a right half-plane.



Causal + Stable: All poles in $\{s : Re\{s\} < 0\}$



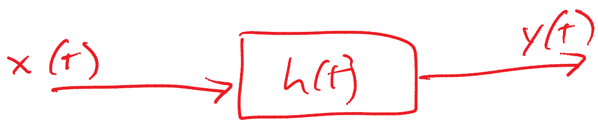


$$y''(t) = x(t) + (2y'(t) + 3y(t))$$

$$y''(t) - 2y'(t) - 3y(t) = x(t)$$

$$s^2 Y(s) - 2sY(s) - 3Y(s) = X(s)$$

$$Y(s)(s^2 - 2s - 3) = X(s)$$

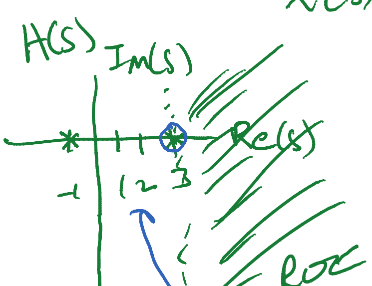


$$(x * h)(t) = y(t)$$

$$X(s)H(s) = Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s^2 - 2s - 3)} =$$

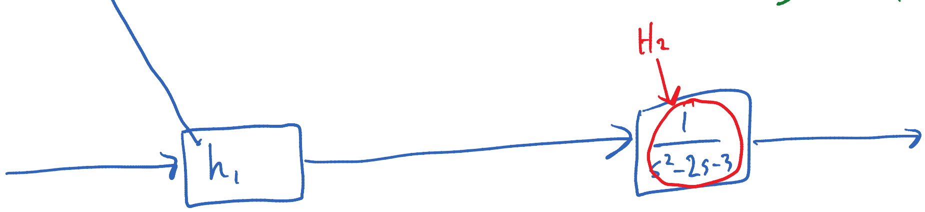


s s.t. $D(s) = 0$

$$D(s) = s^2 - 2s - 3 = (s-3)(s+1) = 0$$

$$\Rightarrow s = 3$$

$$s = -1$$



$$\frac{\cancel{(s-z_1)}(s-z_2)}{(s-p_1)\cancel{(s-z_1)}}$$

$\rightarrow$ p. & zero at $s = 3$ (in H_1)

$\overline{(s-p_1)\cancel{(s-z_1)}}$

LTI: Put zero at $s=3$ (in H_1)

$$H(s) = H_1(s)\,H_2(s)$$

Let $H_1(s) = s-3$

$$h_1(t) = \delta'(t) - 3\delta(t)$$

Partial Fraction Expansion:

$$H(s) = \frac{1}{(s-3)(s+1)}$$

$$= \frac{A}{s-3} + \frac{B}{s+1}$$

Objective:

$$\frac{N(s)}{(s-p_1)^{K_1}(s-p_2)^{K_2}\cdots} = \frac{A_{11}}{(s-p_1)} + \frac{A_{12}}{(s-p_1)^2} \cdots \frac{A_{1K_1}}{(s-p_1)^{K_1}} + \frac{A_{21}}{s-p_2} - - -$$

Assume $N(s)$ is lower order than $D(s)$

Example:

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$1 = \cancel{A(s+1)} + B(s-3) \leftarrow \text{Try } s=-1$$

$$= (A+B)s + (A-3B)$$

$1 = -4B$

$\Rightarrow B = \frac{-1}{4}$

Try $s=3$:

$1 = 4A$

$A = \frac{1}{4}$

$$\Rightarrow \quad 1 = A - 3B$$
$$0 = A + B$$

$$\frac{1}{s^2-2s-3} = \frac{1}{(s-3)(s+1)} = \frac{\frac{1}{4}}{s-3} - \frac{\frac{1}{4}}{s+1}$$

Check

$$\frac{\frac{1}{4}(s+1) - \frac{1}{4}(s-3)}{(s-3)(s+1)} = 1$$

Example:

$$\frac{2s^2 - s + 5}{s^2 + 2s + 1} = 2 + \frac{-5s+3}{s^2+2s+1}$$

$$= 2 + \frac{-5s+3}{\ldots}$$

s^2+2s+1

$$\begin{array}{r|l} & \;\; 2 \\ s^2+2s+1 & 2s^2 - s + 5 \\ & -\left(2s^2 + 4s + 2\right) \\ \hline & \quad -5s + 3 \end{array}$$

$$= 2 + \frac{-5s+3}{(s+1)^2}$$

$$= 2 + \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$B = 8$$

$$\xrightarrow{\mathcal{L}^{-1}} 2\delta(t) + Ae^{-t}u(t) + 8te^{-t}u(t)$$