Lecture 17

Tuesday, November 22, 2011
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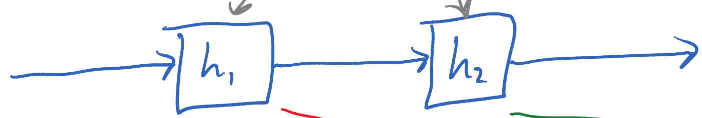
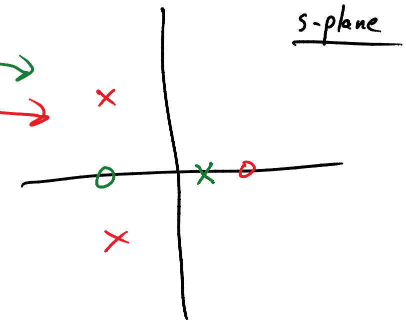
Cascaded Systems:

Both rational Laplace transforms

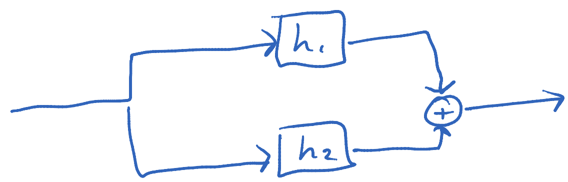




Poles and zeros combine.

$$H_1(z)\,H_2(z) = \frac{N_1(z)\,N_2(z)}{D_1(z)\,D_2(z)}$$

Parallel Systems:



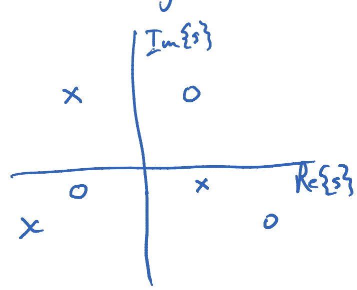
Poles combine:

$$H_1(z) + H_2(z) = \frac{N_1(z)}{D_1(z)} + \frac{N_2(z)}{D_2(z)} = \frac{N_1(z)D_2(z) + N_2(z)D_1(z)}{D_1(z)D_2(z)}$$

System Design and Geometry

All-Pass Filter:

$|H(s)| = 1$ for $s = j\omega$



- Not real
- Either:
  - Causal and unstable
  - or Two-sided and stable
  - or Two-sided and unstable
  - or anti-causal and unstable

Inverse Systems:

$$x(t) \longrightarrow \boxed{h} \longrightarrow \boxed{h^{-1}} \longrightarrow x(t)$$

Change poles to zeros and zeros to poles. (and invert scaling factor)

Can a causal, stable system have a causal, stable inverse?

- Examples: Systems with zeros

(Causal and stable = All poles in left-half plane)

Yes

If all poles and zeros are in left-half plane
(This will often involve a derivative in one of the systems).

- Such systems (or signals) are called minimum-phase

Magnitude^2 of Fourier Transform:

$|H(jw)|^2$ using pole-zero locations.

Attempt 1:
$$|H(s)|^2 = H(s)\,H^*(s)$$
$$= \frac{N(s)\;N^*(s)}{D(s)\;D^*(s)}$$
$$= |C|^2 \frac{(s-z_1)\cdots(s-z_K)(s^*-z_1^*)\cdots(s^*-z_K^*)}{(s-p_1)\cdots(s-p_m)(s^*-p_1^*)\cdots(s^*-p_m^*)}$$

Not polynomials in s.
Can't use our geometry tricks.

Attempt 2: We only care to evaluate $|H(s)|^2$ for $s=jw$

$\Rightarrow$ $s^* = -s$, when $s=jw$.

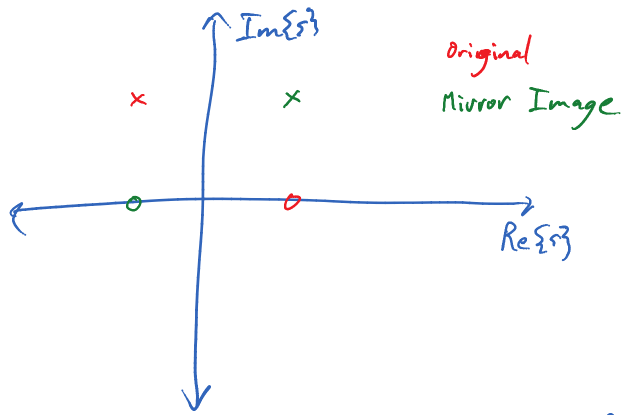
$\Rightarrow s^* = -s$, when $s = j\omega$.

$\Rightarrow$ For $s = j\omega$: $$|H(s)|^2 = \frac{|c|^2 (s-z_1)\cdots(s-z_K)(-s-z_1^*)\cdots(-s-z_K^*)}{(s-p_1)\cdots(s-p_m)(-s-p_1^*)\cdots(-s-p_m^*)}$$

Where are poles and zeros?

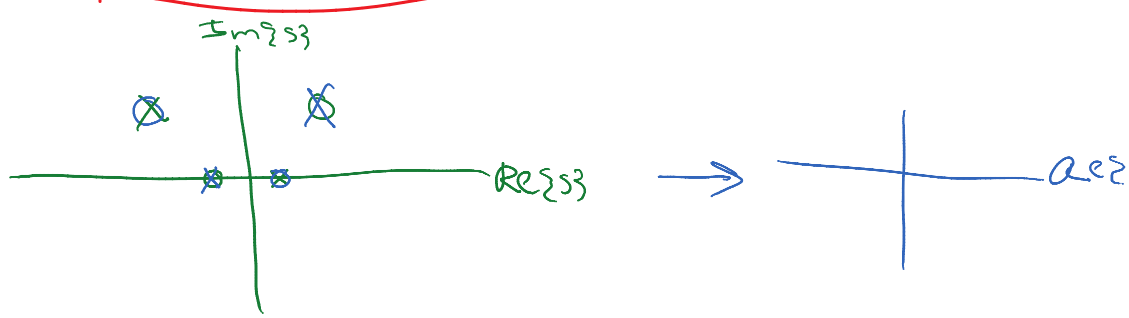
Poles are $p_1 \cdots p_m \; -p_1^* \cdots -p_m^*$ (Flip poles and zeros across imaginary axis)

Zeros are $z_1 \cdots z_K \; -z_1^* \cdots -z_K^*$



Fourier transform of combined system is FT-squared of original system.

All pass Filter example:



Butterworth Filter (9.7.5):

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

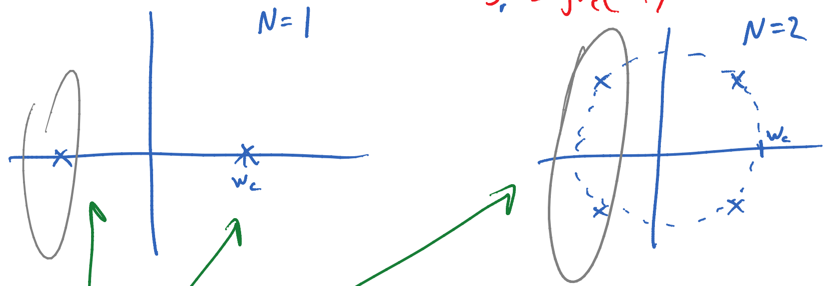
Let's write as $$\frac{1}{1 + (s/j\omega_c)^{2N}}$$ ← Polynomial in s ← Polynomial in s

Let's write as $\dfrac{1}{1+\left(\frac{s}{j\omega_c}\right)^{2N}}$ ← Polynomial in s

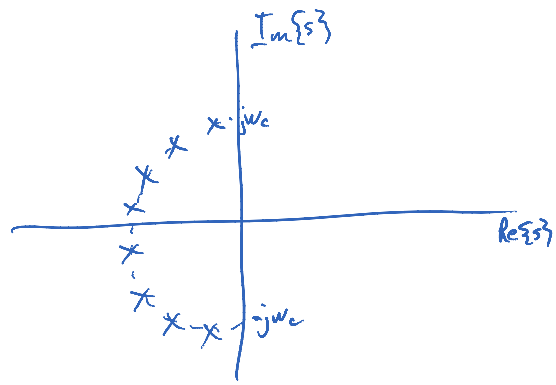
Where are the poles?

$$1+\frac{s^{2N}}{(j\omega_c)^{2N}}=0$$

$$\Rightarrow \quad s_i = j\omega_c(-1)^{\frac{1}{2N}}$$



Select half of the poles and zeros, one from each pair across the imaginary axis.



Z-transform (Discrete-time)

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$$

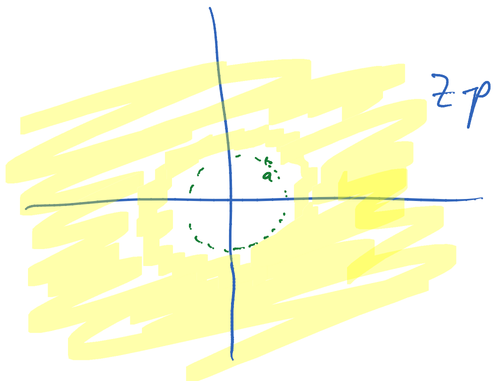
Example: $x[n]=a^n u[n]$

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n]\,z^{-n}$$

$\left|\frac{a}{z}\right|<1$

$$=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \begin{cases}\dfrac{1}{1-\frac{a}{z}}, & |z|>|a|\end{cases}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \begin{cases} \dfrac{1}{1-\frac{a}{z}}, & |z|>|a| \\ \text{doesn't converge}, & |z|\le |a| \end{cases}$$



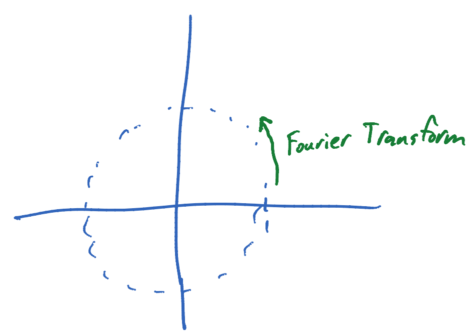
In general, ROC is a ring: $a < |z| < b$

DTFT:

$$X(f) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x[n] \left(e^{j2\pi f}\right)^{-n}$$

$$= \underbrace{X\left(e^{j2\pi f}\right)}_{z\text{-transform}}$$

Notice $|z| = |e^{j2\pi f}| = 1$



– Stable if ROC contains to unit circle.

Properties:

Delay: $x[n] \longrightarrow X(z)$

$x[n-k]$

$$\sum_{n=-\infty}^{\infty} x[n-k]\, z^{-n} = \sum_{m=-\infty}^{\infty} x[m]\, z^{-(m+k)} = z^{-k} X(z)$$

Let $m = n-k$