Lecture 7
ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

Fall 2011-12

Introduction to Fourier Transforms

- Fourier transform as a limit of the Fourier series
- Inverse Fourier transform: The Fourier integral theorem
- Example: the rect and sinc functions
- Cosine and Sine Transforms
- Symmetry properties
- Periodic signals and δ functions

Fourier Series

Suppose $x(t)$ is not periodic. We can compute the Fourier series as if x was periodic with period T by using the values of $x(t)$ on the interval $t \in [-T/2, T/2)$.

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi k f_0 t}\,dt, \\ x_T(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}, \end{aligned}$$

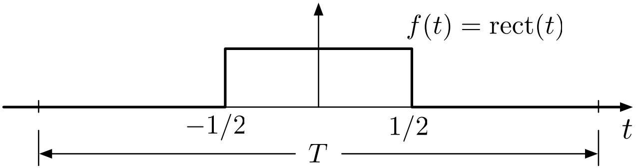
where $f_0 = 1/T$.

The two signals x and x_T will match on the interval $[-T/2, T/2)$ but $\tilde{x}(t)$ will be periodic.

What happens if we let T increase?

Rect Example

For example, assume $x(t) = \mathrm{rect}(t)$, and that we are computing the Fourier series over an interval T,



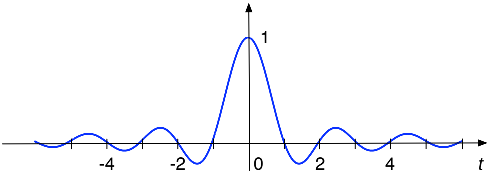
The fundamental period for the Fourier series in T, and the fundamental frequency is $f_0 = 1/T$.

The Fourier series coefficients are

$$a_k = \frac{1}{T}\,\mathrm{sinc}\left(kf_0\right)$$

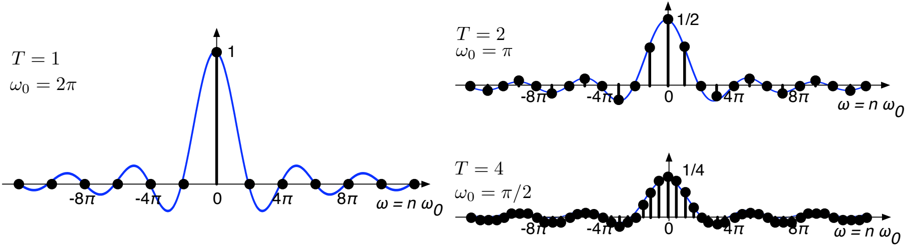
where $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

The Sinc Function



Rect Example Continued

Take a look at the Fourier series coefficients of the rect function (previous slide). We find them by simply evaluating $\frac{1}{T}\,\mathrm{sinc}(f)$ at the points $f = kf_0$.



More densely sampled, same sinc() envelope, decreased amplitude.

Fourier Transforms

Given a continuous time signal $x(t)$, define its *Fourier transform* as the function of a real f:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

This is similar to the expression for the Fourier series coefficients.

Note: Usually $X(f)$ is written as $X(i2\pi f)$ or $X(i\omega)$. This corresponds to the Laplace transform notation which we encountered when discussing transfer functions $H(s)$.

We can interpret this as the result of expanding $x(t)$ as a Fourier series in an interval $[-T/2, T/2)$, and then letting $T \to \infty$.

The Fourier series for $x(t)$ in the interval $[-T/2, T/2)$:

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi k f_0 t}\,dt.$$

Define the truncated Fourier transform:

$$X_T(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi f t}\,dt$$

so that

$$a_k \;=\; \frac{1}{T}X_T(kf_0) \;=\; \frac{1}{T}X_T\left(\frac{k}{T}\right).$$

The Fourier series is then

$$x_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T(kf_0) e^{j2\pi k f_0 t}$$

The limit of the truncated Fourier transform is

$$X(f) = \lim_{T\to\infty} X_T(f)$$

The Fourier series converges to a Riemann integral:

$$\begin{aligned} x(t) &= \lim_{T\to\infty} x_T(t) \\ &= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T\left(\frac{k}{T}\right) e^{j2\pi \frac{k}{T} t} \\ &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df. \end{aligned}$$

Continuous-time Fourier Transform

Which yields the *inversion formula* for the Fourier transform, the *Fourier integral theorem*:

$$\boxed{\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt, \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df. \end{aligned}}$$

Comments:

- There are usually technical conditions which must be satisfied for the integrals to converge – forms of smoothness or Dirichlet conditions.
- The intuition is that Fourier transforms can be viewed as a limit of Fourier series as the period grows to infinity, and the sum becomes an integral.
- $\int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$ is called the *inverse Fourier transform* of $X(f)$. Notice that it is identical to the Fourier transform except for the sign in the exponent of the complex exponential.
- If the inverse Fourier transform is integrated with respect to ω rather than f, then a scaling factor of $1/(2\pi)$ is needed.

Cosine and Sine Transforms

Assume $x(t)$ is a possibly complex signal.

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(t) \left(\cos(2\pi ft) - j \sin(2\pi ft)\right) dt \\
&= \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t)\, dt.
\end{aligned}
$$

Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $x(t)$ as

$$\mathcal{F}\left[x(t)\right] = X(f)$$

and the inverse Fourier transform of $X(f)$ as

$$\mathcal{F}^{-1}\left[X(f)\right] = x(t).$$

Note that

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[x(t)\right]\right] = x(t)$$

and at points of continuity of $x(t)$.

Duality

Notice that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are almost the same.

Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

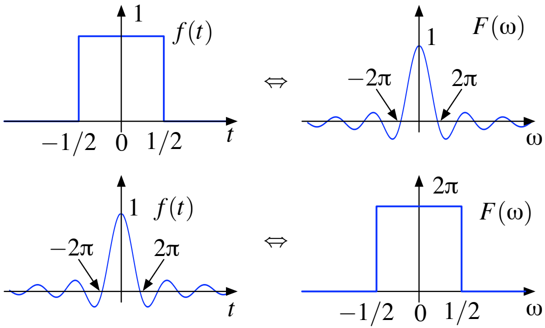
In other words, $\mathcal{F}\left[\mathcal{F}\left[x(t)\right]\right] \ = \ x(-t)$.

Example of Duality

- Since $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$ then

$$\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(-f) = \mathrm{rect}(f)$$

(Notice that if the function is even then duality is very simple)



Generalized Fourier Transforms: δ Functions

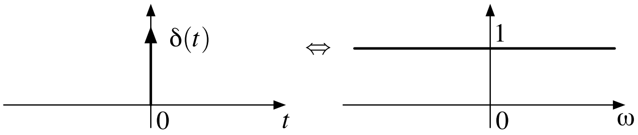
A unit impulse $\delta(t)$ is not a signal in the usual sense (it is a generalized function or distribution). However, if we proceed using the sifting property, we get a result that makes sense:

$$\mathcal{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = 1$$

so

$$\delta(t) \Leftrightarrow 1$$

This is a *generalized Fourier transform*. It behaves in most ways like an ordinary FT.



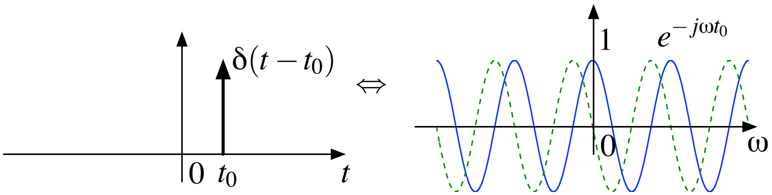
Shifted δ

A shifted delta has the Fourier transform

$$\begin{aligned} \mathcal{F}\left[\delta(t-t_0)\right] &= \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi f t} dt \\ &= e^{-j2\pi t_0 f} \end{aligned}$$

so we have the transform pair

$$\delta(t-t_0) \Leftrightarrow e^{-j2\pi t_0 f}$$



Constant

Next we would like to find the Fourier transform of a constant signal $x(t) = 1$. However, direct evaluation doesn't work:

$$\begin{aligned} \mathcal{F}[1] &= \int_{-\infty}^{\infty} e^{-j2\pi ft} dt \\ &= \left. \frac{e^{-j2\pi ft}}{-j2\pi f} \right|_{-\infty}^{\infty} \end{aligned}$$

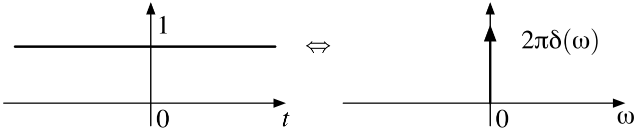
and this doesn't converge to any obvious value for a particular f.

We instead use duality to guess that the answer is a δ function, which we can easily verify.

$$\begin{aligned} \mathcal{F}^{-1}[\delta(f)] &= \int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df \\ &= 1. \end{aligned}$$

So we have the transform pair

$$1 \Leftrightarrow \delta(f)$$



This also does what we expect – a constant signal in time corresponds to an impulse a zero frequency.

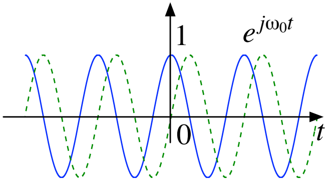
Sinusoidal Signals

If the δ function is shifted in frequency,

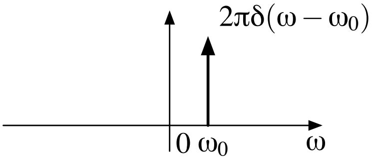
$$\begin{aligned}\mathcal{F}^{-1}\left[\delta(f-f_0)\right] &= \int_{-\infty}^{\infty} \delta(f-f_0) e^{j2\pi ft} df \\ &= e^{j2\pi f_0 t}\end{aligned}$$

so

$$e^{j2\pi f_0 t} \Leftrightarrow \delta(f-f_0)$$



$\Leftrightarrow$



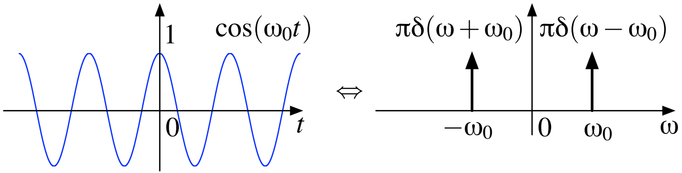
Cosine

With Euler's relations we can find the Fourier transforms of sines and cosines

$$\begin{aligned}
\mathcal{F}\left[\cos(2\pi f_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\right] \\
&= \frac{1}{2}\left(\mathcal{F}\left[e^{j2\pi f_0 t}\right] + \mathcal{F}\left[e^{-j2\pi f_0 t}\right]\right) \\
&= \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right).
\end{aligned}$$

so

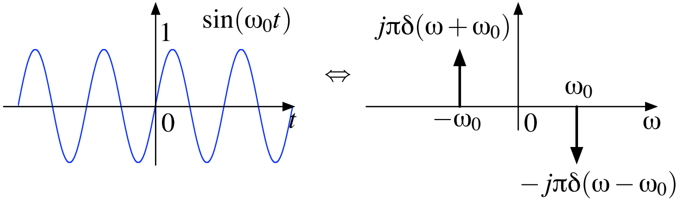
$$\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right).$$



Sine

Similarly, since $\sin(f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t})$ we can show that

$$\sin(f_0 t) \Leftrightarrow \frac{j}{2}\left(\delta(f + f_0) - \delta(f - f_0)\right).$$



The Fourier transform of a sine or cosine at a frequency f_0 only has energy exactly at $\pm f_0$, which is what we would expect.