Lecture 9
ELE 301: Signals and Systems

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Discrete-time Fourier Transform

- Represent a Discrete-time signal using δ functions
- Properties of the Discrete-time Fourier Transform
  - Periodicity
  - Time Scaling Property
  - Multiplication Property
- Periodic Discrete Duality
- DFT
- Constant-Coefficient Difference Equations

Fourier Transform for Discrete-time Signals

$$
\begin{aligned}
x[n] &= \int_1 X(f) e^{i2\pi fn} df, \\
X(f) &= \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi fn}.
\end{aligned}
$$

$X(f)$ is always periodic with period 1.

Continuous-representation of a discrete-time signal

$$x(t) \triangleq \sum_{k=-\infty}^{\infty} x[k]\delta(t-k).$$

Notice that

$$\begin{aligned}
\mathcal{F}[x(t)] &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\delta(t-k) \right) e^{-i2\pi ft} dt \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x[k]\delta(t-k) e^{-i2\pi ft} dt \\
&= \sum_{k=-\infty}^{\infty} x[k] e^{-i2\pi fk} dt.
\end{aligned}$$

Properties of the Discrete-time Fourier Transform

Inherits properties from continuous-time.

Easy Properties:

- Linearity
- Conjugation
- Convolution = Multiplication in frequency domain
- Parseval's Theorem (integrate over one period)
- Time shift

Properties that require care:

- Time-scaling
- Multiplication (circular convolution in frequency)

Time-scaling

In continuous time we can scale by an arbitrary real number. In discrete-time we scale only by integers.

For an integer k, define

$$x_k[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k, \\ 0 & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

$$x_k[n] \Leftrightarrow X(kf).$$

Compressing in time

Compressing in time requires decimation.

Discrete Difference

What is the Fourier transform of $y[n] = x[n] - x[n-1]$?

Dual Derivative Formula

The dual to the continuous-time differentiation formula still holds.

$$nx[n] \Leftrightarrow \frac{i}{2\pi}X'(f).$$

Accumulation

What is the Fourier transform of $y[n] = \sum_{m=-\infty}^{n} x[m]$?

(Hint: Inverse of discrete difference)

Parseval's Theorem

Theorem:

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-1/2}^{1/2} |X(f)|^2 \, df.$$

Multiplication Property

Multiplication in time equates to *c*ircular convolution in frequency.

Notice that multiplying δ functions is not well defined.

This is dual to what we saw in the Fourier series.

Discrete Periodic Duality

periodic signal = discrete transform (though not integer f)

discrete signal = periodic transform

Discrete Fourier Transform

Notice that a discrete and periodic signal will have a discrete and periodic transform. This is convenient for numerical computation (computers and digital systems).

The *DFT* is (almost) equivalent to the discrete-time Fourier series of the periodic extension.

For period N, let

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Then

$$DFT[x] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

The DFT Matrix

Since the Fourier transform is linear, the DFT can be encompassed in a matrix.

$$DFT[x] = Fx.$$

Matlab uses the fast-Fourier-transform algorithm to compute the DFT (using the *fft* command).

Constant-Coefficient Difference Equations

$$\sum_{k=0}^{n} a_k y[n-k] \; = \; \sum_{k=0}^{M} b_k x[n-k].$$

Find the Fourier Transform of the impulse response (the transfer function of the system, $H(f)$) in the frequency domain.